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SOME EXTENSIONS OF MITROVIC'S METHOD  
IN ANALYSIS AND DESIGN OF  
FEEDBACK CONTROL SYSTEMS

HO HYON CHOE

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SOME EXTENSIONS OF MITROVIC'S METHOD  
IN ANALYSIS AND DESIGN  
OF FEEDBACK CONTROL SYSTEMS

by

Ho Hyon Choe  
/)

Lieutenant, Republic of Korea Navy

Submitted in partial fulfillment of  
the requirements for the degree of

MASTER OF SCIENCE  
IN  
ELECTRICAL ENGINEERING

United States Naval Postgraduate School  
Monterey, California

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the thesis requirements for the degree of  
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United States Naval Postgraduate School



## ABSTRACT

The basic Mitrovic's method for analysis and design of linear feedback control system is studied. Some further properties of the basic Mitrovic's equations and curves are investigated and applied to analysis and design of linear feedback control systems. The method developed in this paper gives more flexibility to the basic Mitrovic's method.

The writer wishes to express his appreciation for the assistance and guidance given him by Dr. G. J. Thaler of the U. S. Naval Post-graduate School in this investigation.



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## 1. Introduction.

The Mitrovic's method is an analysis and design technique of linear feedback control systems using algebraic equations and graphical methods. The method is based on conformal mapping of the S-plane into the real domain through the characteristic equation of a feedback control system, and the basic principles involved in the Mitrovic's method are derived by applying the basic theorem by Cauchy in the mapping procedure.

The fundamental theories are adequately developed and applications of the method are demonstrated in references 1 and 2. The main advantages of the Mitrovic's method seem to lie in the fact that all operations are carried out in the real domain and that it has potentiality of controlling or adjusting multiple variable parameters.

The analysis of a system in the Mitrovic's method is done by analyzing the relative location of the Mitrovic's working point M with respect to Mitrovic's curves, and design of a system is to reshape the configuration of Mitrovic's curves and the M point associated with the system of interest so as to meet the design specifications. The analysis and design techniques are deduced from the fundamental properties of the  $B_0$  vs  $B_1$  curves for  $\zeta = \zeta_1$  denoted by  $\sqrt{\zeta_1}$  with the M point as listed below.

- 1) If the M point is located at  $\omega_n = \omega_{n1}$  on the  $B_0$  vs  $B_1$  curve for  $\zeta = \zeta_1$  then the characteristic equation associated with the  $\sqrt{\zeta_1}$  curve and M point has a pair of complex conjugate roots  $\tau = -\zeta_1 \omega_{n1} \pm j\sqrt{1-\zeta_1^2} \omega_{n1}$
- 2) If tangent lines are drawn from the M point to the  $B_0$  vs  $B_1$  curve for  $\zeta = 1.0$  (the  $\sqrt{1.0}$  curve), then the negatives of the slopes of the lines are the real roots of the characteristic equation.



- 3) If the M point is located on a region enclosed by a  $\zeta_1$  curve encircling in the clockwise direction, and if the M point is in the first quadrant on the  $B_0$  vs  $B_1$  plane, then all of the roots of the characteristic equation have  $\zeta$  greater than  $\zeta_1$ .

In this paper further properties of Mitrovic's equations and curves on the  $B_0$  vs  $B_1$  plane are investigated to extend the method to problems which are not treated in the literature, and to give the method a better flexibility.

In Section 2 the  $\zeta_0$  curves are studied to provide a procedure to evaluate frequency response of closed and open loop systems on the  $B_0$  vs  $B_1$  plane so that one can obtain information for frequency response and time response simultaneously. Constant band width curves are introduced on the  $B_0$  vs  $B_1$  plane to design a system to meet the band width specification precisely with other specifications such as  $\zeta$  and  $K_v$  specifications.

In Section 3 fourth order system curves are studied to provide fourth order system charts to fit any fourth order system. The analysis and design procedures developed in references 1 and 2 can be worked out with preconstructed curves on the charts and also a different approach of design technique is tried with fourth order charts.

In Section 4 an analytical design technique is derived so that the labor required in constructing Mitrovic's curves can be avoided or minimized.

Thus each section can be regarded as independent from another; however, the principles developed in Section 2 may go with any of the sections to give information on frequency response.



2. Frequency responses of closed and open loop systems on the Mitrovic's  $B_0$  vs  $B_1$  plane.

Mitrovic's curves for  $\zeta = 0$ , the  $\Gamma_0$  curves, are obtained by mapping the imaginary axis of the S-plane in to the Mitrovic's plane.

For a characteristic equation of a system

$$F(s) = s^n + a_{n-1}s^{n-1} + \dots + a_1s + a_0 = 0 \quad (2-1)$$

Mitrovic's  $B_0$  vs  $B_1$  curve for  $0 \leq \zeta \leq 1$  is defined by

$$\begin{aligned} B_1 &= \sum_{k=2}^n a_k \phi_k(\zeta) \omega_n^{k-1} \\ B_0 &= \sum_{k=2}^n -a_k \phi_{k-1}(\zeta) \omega_n^k \end{aligned} \quad (2-2)$$

where  $\omega_n$  and  $\zeta$  are the natural frequency and damping ratio respectively of a root of the closed loop system, and

$\phi_k(\zeta)$  is a function of  $\zeta$  only. (See Appendix I.)

which is a curve on the Mitrovic's  $B_0$  vs  $B_1$  plane independent of the last two coefficients  $a_0$  and  $a_1$  of the characteristic Equation (2-1).

Mitrovic's working point M on the  $B_0$  vs  $B_1$  plane is defined by the coefficients  $a_1$  and  $a_0$  such that

$$M = M(a_1, a_0)$$

thus, for a given characteristic equation, a set of  $B_0$  vs  $B_1$  curves and an M point are uniquely determined and conversely.

For a particular value of damping coefficient  $\zeta = 0$  the equation defining the  $\Gamma_0$  curve is, (using proper values of  $\phi_k(0)$  and replacing  $\omega_n$  by  $\omega$  the frequency on imaginary axis of the S-plane in Equation (2-2)),

$$\begin{aligned} B_1 &= a_3 \omega^2 - a_5 \omega^4 + a_7 \omega^6 - \dots \\ B_0 &= a_2 \omega^2 - a_4 \omega^4 + a_6 \omega^6 - \dots \end{aligned} \quad (2-3)$$



Because of the nature of the  $\Gamma_0$  curve, the frequency dependence of Equation (2-1) can be represented by the lengths of vectors from the M point to the  $\Gamma_0$  curve. This indicates that frequency response of a closed loop system can be evaluated on the Mitrovic's plane.

In this section relations between the frequency response (magnitude and phase) of a closed loop system and the  $\Gamma_0$  curve with the M point on the Mitrovic's  $B_0$  vs  $B_1$  plane are investigated. Band width of a closed loop system is evaluated and the locus of the M points for a system to meet the specified band width is constructed.



## 2-1. Interpretation of equations (magnitude of frequency response) on the $B_0$ vs $B_1$ plane.

Consider a closed loop transfer function of a feedback control system

$$\frac{C}{R}(S) = \frac{N(S)}{F(S)} = \frac{N(S)}{S^n + a_{n-1}S^{n-1} + \dots + a_1S + a_0} \quad (2-4)$$

where  $N(S)$  is a polynomial of  $S$  of order less than  $n$ .

Note that  $F(S) = 0$  is the characteristic equation from

which the associated  $\Gamma_0$  curve and  $M$  point are determined.

The magnitude of the frequency response of the system represented by Equation (2-4) is

$$A = \left| \frac{C}{R}(j\omega) \right| = \frac{|N(j\omega)|}{|\operatorname{Re}\{F(j\omega)\} + j\operatorname{Im}\{F(j\omega)\}|} \quad (2-5)$$

$\operatorname{Re}\{F(j\omega)\}$  and  $\operatorname{Im}\{F(j\omega)\}$  can be represented in terms of  $a_0$ ,  $B_0$  and  $a_1$ ,  $B_1$  respectively where  $B_1$  and  $B_0$  are the Mitrovic's variables defining the  $\Gamma_0$  curve which are functions of  $\omega$  defined by Equation (2-3).

Rewriting  $F(S)$  in Equation (2-4) as the sum of even  $F(S)$  and odd  $F(S)$ ,

$$F(S) = (a_0 + a_2S^2 + a_4S^4 + \dots) + S(a_1 + a_3S^2 + a_5S^4 + \dots)$$

$$\begin{aligned} \text{then } F(j\omega) &= \operatorname{Re}\{F(j\omega)\} + j\operatorname{Im}\{F(j\omega)\} \\ &= [a_0 - (a_2\omega^2 - a_4\omega^4 + \dots)] + j\omega[a_1 - (a_3\omega^2 - a_5\omega^4 + \dots)] \end{aligned}$$

therefore by using Equation (2-3)

$$\begin{aligned} \operatorname{Re}\{F(j\omega)\} &= a_0 - (a_2\omega^2 - a_4\omega^4 + \dots) = a_0 - B_0 \\ \operatorname{Im}\{F(j\omega)\} &= \omega[a_1 - (a_3\omega^2 - a_5\omega^4 + \dots)] = \omega(a_1 - B_1) \end{aligned}$$

thus, Equation (2-5) can be written as

$$A = \left| \frac{C}{R}(j\omega) \right| = \frac{|N(j\omega)|}{|(a_0 - B_0) + j\omega(a_1 - B_1)|} \quad (2-6)$$



The interpretation of Equation (2-6) in  $B_0$  vs  $B_1$  plane is as follows. The magnitude  $A$  at frequency  $\omega$  is the ratio of the length of two vectors  $N(j\omega)$  and  $(a_0 - B_0) + j\omega(a_1 - B_1)$ .

Since  $a_1$  and  $a_0$  are the coordinates of the point  $M(a_1, a_0)$  on the  $B_0$  vs  $B_1$  plane, and  $B_1$  and  $B_0$  at any frequency are determined by the coordinate of a point at  $\omega$  on  $\Gamma_0$ , the numerical value of the denominator of Equation (2-6) can be determined easily.

Suppose the  $\Gamma_0$  curve and  $M$  point configuration is as shown in Fig. 2-1 and choose a point  $P$  on the  $\Gamma_0$  curve at which the frequency is  $\omega$ , then the coordinates of  $P$  are  $(B_1(\omega), B_0(\omega))$ . Locate a point  $Q$  at  $(a_1, B_0(\omega))$  so that  $Q$  is at the same ordinate as  $P$  and is on the line  $B_1 = a_1$ , then the distances from  $M$  to  $Q$  and  $Q$  to  $P$  are

$$\overline{MQ} = |a_0 - B_0(\omega)|, \quad \overline{QP} = |a_1 - B_1(\omega)| \quad \text{respectively.}$$

Locate a point  $R$  on the line segment  $QP$  such that  $\overline{QR} = \omega \cdot \overline{QP}$  and associate a frequency  $\omega$  to the point  $R$ , then  $\overline{QR} = \omega \cdot \overline{QP} = \omega |a_1 - B_1(\omega)|$  and  $\overline{MR} = (\overline{MQ}^2 + \overline{QR}^2)^{1/2} = |(a_0 - B_0(\omega)) + j\omega(a_1 - B_1(\omega))|$  which is the value of the denominator of Equation (2-6) at frequency  $\omega$ .

As  $\omega$  varies, the point  $P$  moves along the  $\Gamma_0$  curve and the point  $Q$  moves along the  $B_1 = a_1$  line maintaining the same ordinate as  $P$ . The point  $R$  will describe a curve which is represented by the dotted curve and denoted by the  $\omega(a_1 - B_1)$  plot. The  $\omega(a_1 - B_1)$  plot is a well defined curve if the  $\Gamma_0$  curve is well defined and has the following common properties for all systems:

- 1) It has frequency on the curve as a parameter such that two points constituting a pair of points one on the  $\Gamma_0$  curve and the other on the  $\omega(a_1 - B_1)$  plot have the same frequencies with the same ordinates.



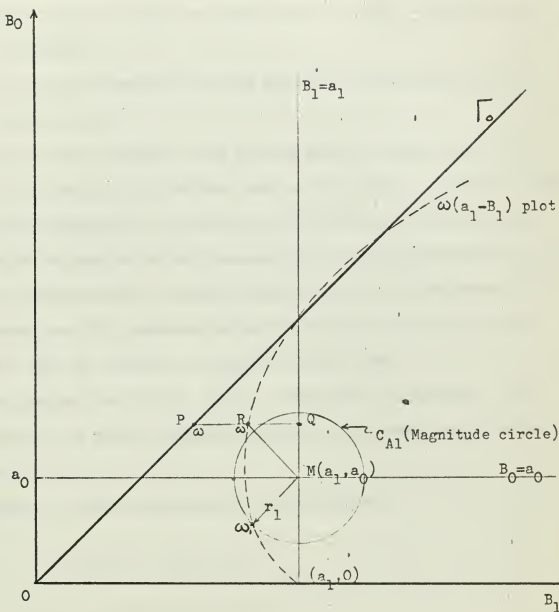


Fig. 2-1 Illustration of obtaining Frequency response for a system with no zeros.



- 2) It starts (or the originating point is) at  $(a_1, 0)$  and passes through the intersection point of the  $\Gamma_0$  curve and the  $B_1 = a_1$  line.
- 3) The points for  $\omega = 1$  on the  $\Gamma_0$  curve and  $\omega(a_1 - B_1)$  plot are identical.
- 4) It lies between the  $\Gamma_0$  curve and the  $B_1 = a_1$  line for  $\omega < 1$  and dually.

If any circle centered at the point M passes through a point at  $\omega = \omega_1$  on the  $\omega(a_1 - B_1)$  plot with radius of  $r_1$  then  $r_1$  represent the value of the denominator in Equation (2-6) at  $\omega = \omega_1$ . The circle centered at the M point is called the magnitude circle. The numerator of Equation (2-6) can not be evaluated from the  $\Gamma_0$  curve as the denominator. It is possible to draw separate Mitrovic's curve for  $N(S)$  and obtain  $N(j\omega)$  but this is labirious if  $N(S)$  is of high order.

The simplest case is for  $N(j\omega)$  independent of frequency. This is the case for a system that has no zeros in its closed loop transfer function.

Suppose a closed loop system has no zero, then

$$A = \frac{a_0}{|(\bar{a}_0 - B_0) + j\omega(\bar{a}_1 - B_1)|} \quad \text{or}$$

$$|(\bar{a}_0 - B_0) + j\omega(\bar{a}_1 - B_1)| = a_0 / A$$

The magnitude  $A_1$  at  $\omega = \omega_1$  is  $a_0/r_1$  and conversely if a magnitude  $A = A_1$  is given then the frequency to give  $A = A_1$  is found by drawing a magnitude circle with radius equal to  $a_0/A_1$ , and locating the point or points where the circle intersects with  $\omega(a_1 - B_1)$  plot.

Some particular points in the frequency response of a closed loop



system such as the resonant point, zero db point and half power point (band width) can be found very simply.

Drawing a magnitude circle tangent to the  $\omega(a_1 - B_1)$  plot gives the resonance point. Magnitude circles with radii of  $a_0$  and  $\sqrt{2} a_0$  give the 0 db point and band width respectively.

For a system with  $N(S)$  in Equation (2-4) of order other than zero, the procedure is slightly modified but still the basic idea is the same.

This is illustrated with Fig. 2-2. Modification must be made to take care of the frequency dependence of the numerator in Equation (2-6). Since the denominator in the equation defining the magnitude of frequency response bears the same form regardless of the order of  $N(S)$ , the construction of the  $\omega(a_1 - B_1)$  plot is the same for all systems.

Rewrite Equation (2-6) as

$$|(a_0 - B_0) + j\omega(a_1 - B_1)| = \frac{1}{A} \cdot |N(j\omega)| \quad (2-7)$$

and interpret as follows:

The quantity at any  $\omega$  on the left side of Equation (2-7) is determined by drawing a magnitude circle, and it is represented by the length of radius of the magnitude circle. On the right side of Equation (2-7),  $|N(j\omega)|$  has some finite value for every  $\omega$ . Then a number which multiplies to  $|N(j\omega)|$  to give the equality in Equation (2-7) is the reciprocal of the magnitude.

In Fig. 2-2, the value of  $|N(j\omega)|$  is represented on a line drawn from M point in an arbitrary direction. This line is denoted by  $|N(j\omega)|$  and at each point on the line, there associate  $\omega$  such that if the point N is at  $\omega$  on the line then

$$\overline{MN} = |N(j\omega)| \quad \text{and} \quad A = \overline{MN} / r$$



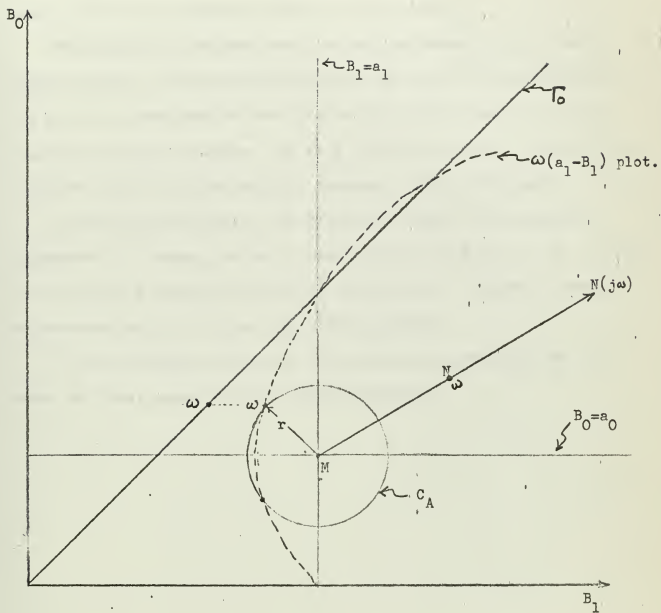


Fig. 2-2 Illustration for obtaining frequency response  
for a system with zeros.



Conversely,  $\omega = \omega_1$  for given magnitude  $A = A_1$  is determined by searching such  $\omega$  that for two magnitude circles with radii of  $r$  and  $Ar$  cross the  $\omega(a_1 - B_1)$  plot and the  $|N(j\omega)|$  line respectively at the same frequency. This can be done by a few tries and errors.

The arguments developed above are not restricted by the order of system equation. The only restriction is to have  $\Gamma_0$  curves plotted on the  $B_0$  vs  $B_1$  plane with the same scales of  $B_1$  and  $B_0$  axes to provide simplest graphical solutions.  $B_0$  vs  $B_1$  curves can always be plotted on such axes scales by using suitable frequency scaling technique.

As mentioned previously, the higher the order of the numerator polynomial of a system, the more complicated the method is. The problem is to provide a convenient method to represent the frequency dependence of the numerator of a closed loop system equation.

In the following paragraphs, such methods are presented for the cases of first, second and third order numerators.



## 2-2. Representation of frequency dependence of $N(S)$ on the $B_0$ vs $B_1$ plane.

In the previous section, it is seen that the frequency dependence of the numerator polynomial  $N(S)$  in a closed loop system equation must be represented as a distance from the M point.

In Fig. 2-3 it is illustrated for  $N(S)$  of first order. When a closed loop system has one zero then  $N(S)$  is of the first order as

$$N(S) = b_1 S + a_0 \quad \text{and} \quad |N(j\omega)| = |a_0 + j b_1 \omega|$$

The real term of  $N(j\omega)$  is independent of frequency. By constructing a rectangle with a fixed side of length equal to  $a_0$  and another side which is in right angle with the fixed side varying linearly with  $\omega$ ,  $|N(j\omega)|$  can be represented by the length of the diagonal.

In Fig. 2-3, a point  $T_2$  is located on the  $B_0 = a_0$  line such that  $\overline{MT_2} = b_1$ . If the line segment  $MT_2$  is calibrated linearly with  $\omega$  such that  $\omega = 0$  at M and  $\omega = 1.0$  at  $T_2$ , and if the same calibration is extended beyond the point  $T_2$ , then at each point on the line  $B_0 = a_0$  to the right of M, there associate  $\omega$  linearly with the distance from M to each point. For any point  $T_1$  at  $\omega = \omega_1$ ,  $\overline{MT_1} = b_1 \omega_1$ . Noting that  $\overline{TT_1} = a_0$ , the length of diagonal of the rectangle  $MT_1Ta_1$  is equal to

$$\overline{MT} = |a_0 + j b_1 \omega_1| = |N(j\omega)|$$

For a system with two zeros in its closed loop system equation,  $N(S)$  is of second order polynomial as

$$N(S) = b_2 S^2 + b_1 S + a_0 \quad \text{and} \quad |N(j\omega)| = |(a_0 - b_2 \omega^2) + j b_1 \omega|$$

The imaginary term can be represented in exactly the same way as in case of first order  $N(S)$ .

The real term is now also frequency dependent, and can be represented on the  $B_1 = a_1$  line with quadratic calibration.



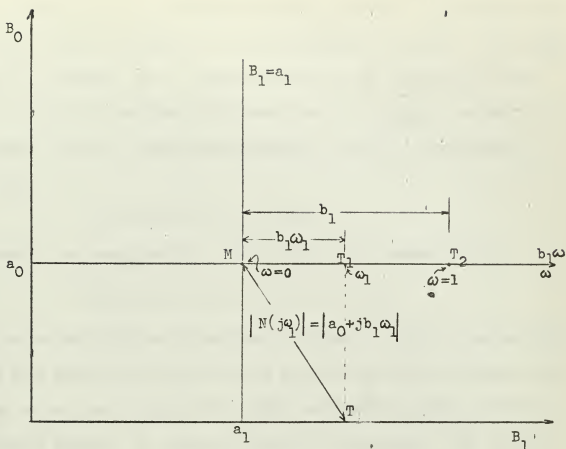


Fig. 2-3 Representation of  $|N(j\omega)| = |a_0 + jb_1\omega|$

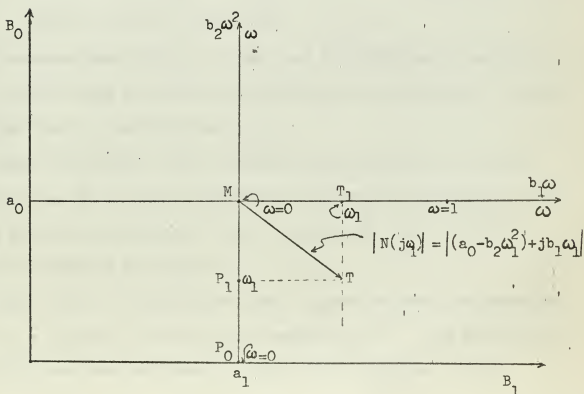


Fig. 2-4 Representation of  $|N(j\omega)| = |(a_0 - b_2\omega^2) + jb_1\omega|$



In Fig. 2-4,  $b_1\omega$  is represented as in Fig. 2-3. To represent the term  $(a_0 - b_2\omega^2)$  the line  $B_1 = a_1$  is calibrated as follows. The point  $P_0(a_1, 0)$  is labeled  $\omega = 0$ . From the point  $P_0$  up each point on the  $B_1 = a_1$  line is calibrated with  $\omega$  such that if a point  $P_1$  on the  $B_1 = a_1$  line reads  $\omega = \omega_1$  then the distance  $\overline{P_0P_1} = b_1\omega_1^2$ . Noting that  $\overline{MP_0} = a_0$ ,

$$\overline{MP_1} = \overline{MP_0} - \overline{P_0P_1} = a_0 - b_2\omega_1^2$$

By forming a rectangle  $MT_1TP_1$ , the length of the diagonal MT is

$$\overline{MT} = |(a_0 - b_2\omega^2) + j b_1\omega| = |N(j\omega)|$$

For a system with  $N(S)$  of third order or higher,  $|N(j\omega)|$  can be represented in a similar way as in case of second order  $N(S)$  with some modifications as the case of first order  $N(S)$  modified to account the second order  $N(S)$ ; however, it requires laborious calculations. For example, for third order  $N(S)$

$$N(S) = b_3S^3 + b_2S^2 + b_1S + a_0 \quad (2-8)$$

$$N(j\omega) = (a_0 - b_2\omega^2) + j(b_1\omega - b_3\omega^3)$$

which indicates that the  $B_0 = a_0$  line must be calibrated to account  $(b_1\omega - b_3\omega^3)$  which requires cubic calculations, and quadratic calculations for fourth order  $N(S)$  and so on.

Those calculations can be avoided by applying Mitrovic's basic equations for the  $\overline{\Gamma}_0$  curve to Equation (2-8) and by proceeding the same way as was done to represent  $F(j\omega)$  by means of  $\omega(a_1 - B_1)$  plot. This is illustrated with Fig. 2-5.

Since  $N(S)$  is of third order, the  $\overline{\Gamma}_0$  curve obtained from Equation (2-8) is a straight line and this is denoted by  $\overline{\Gamma}_{0N}$ . Let Mitrovic's variables associated with Equation (2-8) be  $B_{1N}$  and  $B_{0N}$ . Then



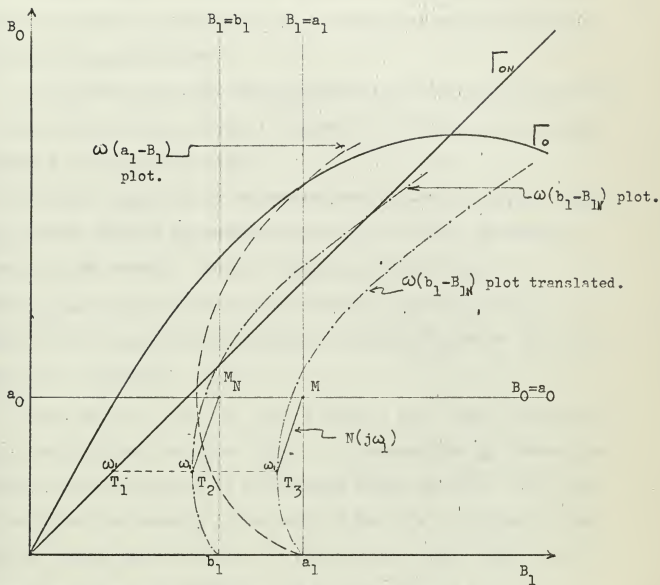


Fig. 2-5 Representation of  $|N(j\omega)| = |(a_0 - B_{0N}) + j\omega(b_1 - B_{1N})|$



$\Gamma_{ON}$  is defined by

$$\begin{aligned} E_{1N} &= b_3 \omega^2 \\ B_{0N} &= b_2 \omega^2 \end{aligned} \quad (2-9)$$

The point  $M_N(b_1, a_0)$  which is analogous to the point  $M(a_1, a_0)$  for the characteristic equation  $F(S) = 0$ , is determined by the coefficients  $b_1$  and  $a_0$  of Equation (2-8).

If  $E_{1N}$  and  $B_{0N}$  axes are chosen identically with  $B_1$  and  $B_0$  axes, then has a slope of  $b_2/b_3$  and  $M_N$  is located on a horizontal line passing through  $M$  on the  $B_0$  vs  $B_1$  plane.

If  $\omega(b_1 - E_{1N})$  plot is constructed from  $\Gamma_{ON}$  and the line  $B_1 = E_{1N} = b_1$ , then by the same reasoning as for  $\omega(a_1 - B_1)$  plot, for any frequency  $\omega_1$  the quantity  $|N(j\omega_1)| = |(a_0 - B_{0N}) + j\omega_1(b_1 - E_{1N})|$  where  $E_{1N}$  and  $B_{0N}$  are coordinates of a point at  $\omega = \omega_1$  on  $\Gamma_{ON}$ , is equal to the distance from  $M_N$  to  $T_2$  where the point  $T_2$  is at  $\omega = \omega_1$  on the  $\omega(b_1 - E_{1N})$  plot.

Since  $|N(j\omega)|$  and  $|(a_0 - B_0) + j\omega(a_1 - B_1)|$  must be compared to get  $A$  the magnitude, and since  $N(j\omega)$  is measured from  $M_N$  whereas the latter is measured from  $M$ , it is necessary either the  $\omega(b_1 - E_{1N})$  plot is translated horizontally to the right or the  $\omega(a_1 - B_1)$  plot to the left so that  $M_N$  and  $M$  coincide. In Fig. 2-5, the  $\omega(b_1 - E_{1N})$  plot is translated to the right by amount of  $a_1 - b_1$ , then

$$|N(j\omega_1)| = \overline{M_N T_2} = \overline{M T_3}$$

The method representing  $|N(j\omega)|$  for  $N(S)$  of up to third order is presented. For high order  $N(S)$ ,  $|N(j\omega)|$  can be represented by an analogous method as in case of third order  $N(S)$ ; however, it is inevitable to be laborious to represent  $|N(j\omega)|$  as the order of  $N(S)$  increases.



On the  $B_0$  vs  $B_1$  plane for feedback control systems with adequate relative stability, the M point is usually located in the region near to the origin as compared with whole region covered by the  $\Gamma_0$  curve. This is due to the fact that  $\Gamma_\zeta$  curves for higher  $\zeta$  are biased towards the origin. Therefore in obtaining frequency response usually a first part of the  $\Gamma_0$  curve is required.

It must be noted that, since the basic Mitrovic's equations are not restricted to the characteristic equation of a feedback control system but apply to any polynomial with positive coefficients, the principles developed above apply to any transfer function which is expressed in the form of a quotient of two polynomials provided the  $B_0$  vs  $B_1$  curve for  $\zeta = 0$  and the M point associated with the transfer function are defined. Therefore it is obvious that the magnitude in the frequency response of an open loop system can be evaluated in a similar way on the same  $B_0$  vs  $B_1$  plane where the magnitude of the closed loop system is evaluated.

In general the open loop frequency response can be obtained by constructing the  $\Gamma_0$  curve and by locating the M point associated with the open loop transfer function and then by proceeding exactly the same way as for the closed loop system. In Section 2-7, some examples for evaluating the phase of the frequency response for both open and closed loop systems are shown, which will well illustrate the above arguments.



2-3. Determination of band width of a closed loop system on the  $B_0$  vs  $B_1$  plane.

For any given system, band width defined at magnitude of  $1/\sqrt{2}$  or half power point can be determined precisely by the method described in Section 2-2.

For a system with no zeros, band width is found at the intersection point of  $\omega(a_1 - B_1)$  plot and a magnitude circle with radius equal to  $\sqrt{2} a_0$ .

For a system with zeros, band width can be found by trying a set of 2 concentric magnitude circles with radii of  $r$  and  $\sqrt{2} r$  such that the two circles cross the  $\omega(a_1 - B_1)$  plot and  $|N(j\omega)|$  respectively at a same value of frequency.

Frequently in the study of feedback control system the point of band width alone of the frequency response is of interest and also sometimes upper bound and/or lower bound of band width is of interest.

By the property of the equation defining band width and associated geometry on the  $B_0$  vs  $B_1$  plane it is possible to set up upper and/or lower bound of band width for a given system. Among the various points in a frequency response curve of a given system, if band width only or approximate value of band width is required, then by setting up the lower and/or upper bound of band width, the effort in plotting  $\omega(a_1 - B_1)$  can be economized or even without plotting any portion of  $\omega(a_1 - B_1)$  the approximate value of band width can be determined.

In this section the three cases for a system with (1) no zeros, (2) with one zero, (3) with two and three zeros are considered.

For a system represented by Equation (2-4), and using Equation (2-6),



the band width defined at half power point is the frequency to satisfy the equation

$$\left| \frac{C}{R}(j\omega) \right| = \frac{|N(j\omega)|}{|(a_0 - B_0) + j\omega(a_1 - B_1)|} = \frac{1}{\sqrt{2}}$$

$$\text{or} \quad |(a_0 - B_0) + j\omega(a_1 - B_1)| = \sqrt{2} |N(j\omega)| \quad (2-10)$$

All that has to be done is to interpret Equation (2-10) on the  $B_0$  vs  $B_1$  plane. The left side of Equation (2-10) can be interpreted by the  $\omega(a_1 - B_1)$  plot as shown in Section 2-1, and for the right side is needed a modification to the method described in Section 2-2 to account for the factor  $\sqrt{2}$ .

Consider a closed loop transfer function with no zero.

Let  $\omega_b$  be the band width, then from Equation (2-10)

$$|(a_0 - B_0(\omega_b)) + j\omega_b(a_1 - B_1(\omega_b))| = \sqrt{2} a_0 \quad (2-11)$$

must hold, where  $B_1(\omega_b)$  and  $B_0(\omega_b)$  are the coordinates of a point on the  $\Gamma_0$  curve at  $\omega = \omega_b$ .

Equation (2-11) implies that a magnitude circle with radius of  $\sqrt{2} a_0$  intersect with the  $\omega(a_1 - B_1)$  plot at  $\omega_b$  which is described in Section 2-2. Since the  $\Gamma_0$  curve tends to infinity in some direction as  $\omega$  tends to infinity,  $\omega(a_1 - B_1)$  plot also tends to infinity. Noting the finite radius of a magnitude circle and that the  $\omega(a_1 - B_1)$  plot starts from a point inside the magnitude circle with radius  $\sqrt{2} a_0$ , it is seen that there exists at least one intersection point between the circle and the  $\omega(a_1 - B_1)$  plot. Depending upon the relative location of the M point with respect to the  $\Gamma_0$  curve and also depending upon the geometric shape of the  $\Gamma_0$  curve there may be more than one such intersection point



implying more than one band width to exist.

On the  $B_0$  vs  $B_1$  plane with the  $\Gamma_0$  curve, by drawing a magnitude circle of radius  $\sqrt{2} a_0$  and by inspecting the values of  $\omega$  at all intersection points of the circle with the  $\Gamma_0$  curve if there are any, and by estimating the shape of the  $\omega(a_1 - B_1)$  plot, information about the band width can be obtained. Also the point at  $\omega = 1$  on  $\Gamma_0$  and the points where the  $\Gamma_0$  curve intersects with  $B_1 = a_1$ ,  $B_0 = a_0$  and  $B_0 = (1 + \sqrt{2})a_0$  lines provide additional information.

Suppose a system is represented by the  $\Gamma_0$  curve and M point as in Fig. 2-6. A magnitude circle  $C_{Ab}$  with radius of  $\sqrt{2} a_0$  is drawn. The points  $P_1, P_2, P_3, P_4$ , and  $P_5$  represent the points where the  $\Gamma_0$  curve intersects with  $C_{Ab}$ ,  $B_0 = a_0$ ,  $B_0 = (1 + \sqrt{2})a_0$  or  $B_1 = a_1$ .

Suppose at each point of those intersections, frequencies are read to be  $\omega_1, \omega_2, \omega_3, \omega_4$  and  $\omega_5$  as shown.

The latter part of the  $\Gamma_0$  curve is neglected since it is assumed that for that part of  $\Gamma_0$ , the  $\omega(a_1 - B_1)$  plot stays outside of  $C_{Ab}$ . In an actual situation this can be judged by looking at the quantity  $\omega_6(a_1 - B_1(\omega_6))$  and  $\omega_7(a_1 - B_1(\omega_7))$  where  $\omega_6$  and  $\omega_7$  are the frequencies at the points where the latter part of  $\Gamma_0$  intersects with  $B_0 = (1 + \sqrt{2})a_0$  and  $B_0 = a_0$  respectively.

The  $\omega(a_1 - B_1)$  plot resulting from the portion of the  $\Gamma_0$  curve above the line  $B_0 = (1 + \sqrt{2})a_0$  can never cross  $C_{Ab}$ , therefore, then there are four possibilities for band width as

$$(1) \quad \omega_3 \leq \omega_b \leq \omega_4$$

$$(3) \quad \omega_1 \leq \omega_b \leq \omega_2$$

$$(2) \quad \omega_2 \leq \omega_b \leq \omega_3$$

$$(4) \quad \omega_b \leq \omega_1$$



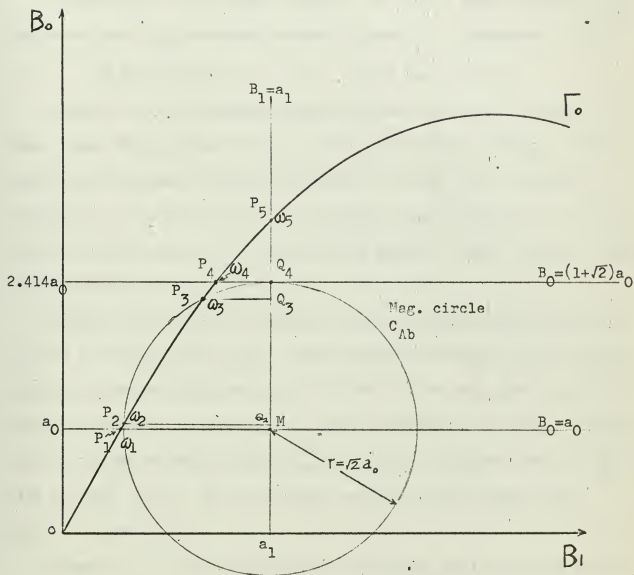


Fig. 2-6 Determining approximate band width  
for a system with no zeros.



Each of the four possibilities above can be evaluated by looking at the point at  $\omega = 1$  on the  $\Gamma_0$  curve.

Suppose  $\omega_3 > 1$  then the  $\omega(a_1 - B_1)$  plot for  $\omega \geq \omega_3$  stays outside of the  $\Gamma_0$  curve in turn outside of  $C_{Ab}$ , indicating  $\omega_b < \omega_3$ .

Suppose  $\omega_3 < 1$  then  $\overline{Q_3P_3} > \omega_3(a_1 - B(\omega_3))$  and the  $\omega(a_1 - B_1)$  plot must cross  $C_{Ab}$  somewhere between  $\omega_3$  and  $\omega_4$  therefore

$$\omega_3 < \omega_b < \omega_4 < 1 \quad \text{or} \quad \omega_3 < \omega_b < 1 < \omega_4$$

Suppose  $\omega = 1$  is located between  $\omega_2$  and  $\omega_3$  or  $\omega_2 < 1 < \omega_3$ , then, since the  $\Gamma_0$  curves for  $\omega_2 < \omega < \omega_3$  are inside of  $C_{Ab}$ , the  $\omega(a_1 - B_1)$  plot must cross  $C_{Ab}$  between  $\omega = 1$  and  $\omega_3$ , therefore  $1 < \omega_b < \omega_3$ . In this case there is still a possibility that  $\omega_b < \omega_3$  but this can be checked by looking at the quantity  $\omega(a_1 - B_1)$  for some  $\omega$  less than  $\omega_2$ .

Suppose  $\omega_1 < 1 < \omega_2$ , then the  $\omega(a_1 - B_1)$  plot stays outside of  $C_{Ab}$  for  $1 \leq \omega < \omega_2$  and  $\omega \geq \omega_3$ . The portion of the  $\omega(a_1 - B_1)$  plot for  $\omega_2 < \omega < \omega_3$  may be inside of  $C_{A2}$ . If this is the case then  $\omega_2 < \omega_b < \omega_3$ , again the possibility that  $\omega_b < \omega_2$  must be checked. If  $\omega(a_1 - B_1)$  for  $\omega_2 < \omega < \omega_3$  has no point inside  $C_{Ab}$  then  $\omega(a_1 - B_1)$  for  $\omega \geq 1$  is outside  $C_{Ab}$  and  $\omega_b < 1$ . If the points  $P_1$  and  $P_2$  are very close, then  $\omega_b \doteq 1 \doteq \omega_1 \doteq \omega_2$ .

Suppose  $\omega_1 > 1$ , then there is no possibility that  $\omega_b$  is in the intervals  $(1, \omega_2)$  and  $\omega \geq \omega_3$ . Since the  $\omega(a_1 - B_1)$  plot at  $\omega = 0$  is at the point  $(a_1, 0)$  and the  $\omega(a_1 - B_1)$  plot at  $\omega = 1$  is outside of  $C_{Ab}$ , band width is definitely  $0 < \omega_b < 1 < \omega_1$  and possibly  $\omega_2 < \omega_b < \omega_3$ .

If either one of the frequencies  $\omega_2$  and  $\omega_3$  is equal to 1, then



$\omega_b = 1$  and other possibilities can be investigated as above.

Thus the four possibilities listed above are analyzed. The geometry involved in Fig. 2-6 is so simple that the relative locations of  $\omega(a_1 - B_1)$  plot for  $\omega \geq \omega_4$ ,  $\omega_3 < \omega \leq \omega_4$ ,  $\omega_2 < \omega \leq \omega_3$ ,  $\omega_1 < \omega \leq \omega_2$  and  $0 \leq \omega \leq \omega_1$  with respect to  $C_{Ab}$  are determined almost by inspection using the point  $\omega = 1$  as a guide, resulting in determination of upper and/or lower bound of band width.

It must be noted that, depending upon the system of interest, there will result various configurations of the  $\Gamma_0$  curve and the working point  $M(a_1, a_0)$ ; however, the basic principle of reasoning is the same as above.

For a system with one zero, the band width defining Equation (2-10) is to have frequency dependent terms in both sides of the equation and requires some modification in the method applied for a system with no zeros. If a system has one zero in its closed loop transfer function, then the equation defining band width is

$$\left| (a_0 - B_0(\omega_b)) + j\omega_b(a_1 - B_1(\omega_b)) \right| = \sqrt{2} \left| a_0 + j b_1 \omega_b \right| \quad (2-12)$$

where  $b_1$  is the coefficient of the  $S^1$  term in the numerator of the given closed loop transfer function.  $B_0(\omega_b)$  and  $B_1(\omega_b)$  are the same as in Equation (2-11).

Suppose the  $\Gamma_0$  curve and M point configuration is as shown in Fig. 2-7. To account for the frequency dependent term  $\sqrt{2} b_1 \omega$  on the right side of Equation (2-12) the line  $B_0 = (1 + \sqrt{2}) a_0$  (or  $B_0 = a_0$  line) is calibrated with  $\omega$  linearly such that the points Q and  $T_1$  correspond to  $\omega = 0$  and  $\omega = 1$  respectively, and the length  $\overline{QT_1}$  is equal to  $\sqrt{2} b_1$ . Then the distance from Q to any point at  $\omega = \omega_1$  on the line







$B_0 = (1 + \sqrt{2})a_0$  will represent  $\sqrt{2} b_1 \omega_1$  and it is obvious that  $\overline{MW} = \sqrt{2} |a_0 + j b_1 \omega'|$  which is the quantity on the right side of Equation (2-12) for  $\omega = \omega'$

Suppose the  $\omega(a_1 - B_1)$  plot is as shown by the dotted curve in Fig. 2-7. Noting the nature of the  $\omega(a_1 - B_1)$  plot that it always starts at the point  $(a_1, 0)$ , crosses  $\Gamma_0$  at  $\omega = 1$  and at the point where  $\Gamma_0$  intersects with  $B_1 = a_1$  line, it is easy to guess or sketch the  $\omega(a_1 - B_1)$  plot.

The band width  $\omega_b$  is some  $\omega$  such that Equation (2-12) holds. This can be interpreted in Fig. 2-7 as follows: Suppose a magnitude circle  $C_{A\omega'}$  is drawn such that it pass through the point W at  $\omega = \omega'$  on the  $B_0 = (1 + \sqrt{2})a_0$  line and suppose it reads  $\omega = \omega''$  on the  $\omega(a_1 - B_1)$  plot. If  $\omega' = \omega''$  then Equation (2-12) holds at frequency equal to  $\omega'$ , therefore  $\omega_b = \omega'$ .

To determine band width one must find a  $C_A$  such that it crosses both  $\omega(a_1 - B_1)$  plot and  $B_0 = (1 + \sqrt{2})a_0$  at the same frequency.

For a  $C_{A\omega'}$  if  $\omega' \neq \omega''$  then the two frequencies are either  $\omega' > \omega''$  or  $\omega' < \omega''$ . By drawing another  $C_{A\omega'}$  until the inequality between  $\omega'$  and  $\omega''$  is reversed from the previous set of  $\omega'$  and  $\omega''$  the bounds of band width can be set up.

For example, for the configuration of curves as shown in Fig. 2-7  $\omega_2 < 1$  is assumed.

For the two magnitude circles  $C_{A0}$  and  $C_{A1}$

	Frequency on $B_0 = (1 + \sqrt{2})a_0$ line		Frequency on $\omega(a_1 - B_1)$ plot
$C_{A0}$	$\omega' = 0$	$<$	$\omega'' = \omega_0$
$C_{A1}$	$\omega' = 1$	$>$	$\omega'' = \omega_1$

in which the inequality sign is reversed, therefore  $0 < \omega_0 < \omega_b < \omega_1 < 1$



It is obvious in Fig. 2-7 that there is no possible band width for  $\omega < \omega_0$ . There may be another band width for  $\omega > \omega_1$ . This can be investigated by trying some magnitude circles.

$C_{A0}$  in Fig. 2-7 is identical to  $C_{Ab}$  in Fig. 2-6. If  $b_1 = 0$  in Fig. 2-7, then  $C_{A\omega'}$  stays on  $C_{A0}$  for all  $\omega'$ , which is the case for a system with no zeros and agrees with Fig. 2-6. It is readily seen from Fig. 2-7 that the effect of zero and forward gain of a system on band width can be analyzed.

For two systems, one for no zero and the other for one zero, if the characteristic equations are the same, then band width of the first system is less than that of the second. This is confirmed in Fig. 2-7 as for the first system  $\omega_b = \omega_0$  and for the second system  $\omega_b > \omega_0$ .

The effect of varying forward gain and the zero on band width for a system with one zero is seen from Fig. 2-7 by writing the numerator of the closed loop transfer function as

$$N(S) = b_1 S + a_0 = KS + KZ$$

Increasing either  $K$  or  $Z$  is to expand both  $C_{A0}$  and  $C_{A1}$  which is to increase  $\omega_0$  and  $\omega_1$  resulting in increased band width. Since  $K$  and  $Z$  do not enter into any coefficient other than  $a_1$  or  $a_0$  of the characteristic equation, the  $\Gamma_0$  curve remains unaffected by either  $K$  or  $Z$ .

Suppose  $K$  is fixed and  $Z$  is variable. Since  $a_1$  and  $b_1$  are independent of  $Z$ , the  $\omega(a_1 - B_1)$  plot and the lines  $B_1 = a_1$ ,  $B_1 = a_1 + \sqrt{2} b_1$  remain unchanged. Varying  $Z$  moves the  $M$  point and the line  $E_0 = a_0$  vertically, thus increasing  $Z$  expands  $C_{A0}$  and  $C_{A1}$  resulting in increased band width.



Suppose  $Z$  is fixed and  $K$  is variable, since  $a_1$ ,  $a_0$  and  $b_1$  are linearly varying with  $K$ , all of the lines in Fig. 2-7 and the  $\omega(a_1 - B_1)$  plot are moved. The  $M$  point is moved along a line with some positive slope and  $C_{A0}$  and  $C_{A1}$  are expanded with increasing  $K$  resulting in increased bandwidth.

In a system which has positive coefficients in both numerator and denominator polynomials of its open loop transfer function,  $b_1$  is always less than  $a_1$  assuming unity feedback. This means that the bandwidth of a system with one zero is always less than that of a system which results from replacing  $b_1$  with  $a_1$ . Therefore for a system whose  $\Gamma_0$  and  $M$  point configuration as shown in Fig. 2-7 bandwidth must be

$$\omega_0 < \omega_b < \omega_2$$

where  $\omega_2$  is the frequency on the  $\omega(a_1 - B_1)$  plot at which the  $\omega(a_1 - B_1)$  plot and magnitude circle  $C_{A2}$  intersect.  $C_{A2}$  is drawn such that it passes through  $T_2$  where  $T_2$  is located on the  $B_0 = (1 + \sqrt{2})a_0$  line at the distance of  $\sqrt{2}a_1$  measured from  $Q$ .

Above inequality is useful to set up the bounds of  $\omega_b$  when  $b_1$  is a variable and when it is required to locate  $M$  point to meet some required bandwidth specification. It must be noted that the bounds determined by the above inequality become poorer as the difference between  $b_1$  and  $a_1$  gets larger.

It is interesting to note that if  $b_1 = a_1$  then the open loop transfer function is of the form, assuming unity feedback,

$$G(s) = \frac{a_1 s + a_0}{s^n + a_{n-1} s^{n-1} + \dots + a_2 s^2}$$

which is a type 2 system, and this system has greater bandwidth than a



system whose open loop transfer function is

$$G(S) = \frac{b_1 S + a_0}{S^n + a_{n-1} S^{n-1} + \dots + a_2 S^2 + (a_1 - b_1) S}$$

For a system with two zeros, bandwidth can be evaluated in a similar way as illustrated in Fig. 2-7. One modification is required in Fig. 2-7 to account for the frequency dependence of  $\sqrt{2} (a_0 - b_2 \omega^2)$ , where  $b_2$  is the coefficient of the  $S^2$  term in the numerator of the closed loop transfer function. This is done by calibrating the line  $B_1 = a_1$  as in Fig. 2-4 to represent  $(a_0 - b_2 \omega_1^2)$  by the distance from M to the point at  $\omega = \omega_1$  on the  $B_1 = a_1$  line. To account for the factor  $\sqrt{2}$ , the origin of calibration or the  $\omega = 0$  point is chosen at a point on the  $B_1 = a_1$  line below the  $B_1$  axis so that it is apart from the M point by a distance of  $\sqrt{2} a_0$ . Then the line  $B_1 = a_1$  is calibrated with  $\omega$  such that any point at  $\omega = \omega_1$  on the line will measure  $\sqrt{2} b_2 \omega_1^2$  by the distance from the  $\omega = 0$  point (originating point) to the point where  $\omega = \omega_1$ .

For a system with 3 zeros, the quantity  $\sqrt{2} |(a_0 - b_2 \omega^2) + j\omega(b_1 - b_3 \omega^2)|$  which is the right side of the bandwidth defining equation can be represented in a similar way as in Fig. 2-5. The only difference is to expand the translated  $\omega(b_1 - B_{1W})$  plot radially from the M point by factor of  $\sqrt{2}$ .

Then for both cases of the above, bandwidth can be evaluated by applying the same analysis as for the system with one zero in Fig. 2-7. The following numerical examples illustrate the arguments developed so far.



### Example 2.1

Obtain frequency response (magnitude) of the closed loop system

$$\frac{C}{R}(s) = \frac{4}{s^3 + 3s^2 + 2s + 4} \quad (2-13)$$

The system represented by Equation (2-13) is a third order system. Since it has no zeros the bandwidth will be less than the frequency at the point where the  $\Gamma_0$  curve and  $B_0 = (1 + \sqrt{2})a_0$  line intersect. If the frequency response is required up to  $\omega = \omega_b$ , the portion of the  $\Gamma_0$  curve needed is below the line  $B_0 = (1 + \sqrt{2})a_0 = 9.65$ .

In Fig. 2-8 the frequency response of the system (2-13) is represented on the  $B_0$  vs  $B_1$  plane.

From the characteristic equation of the system (2-13) the  $\Gamma_0$  curve is defined by

$$\begin{aligned} B_1 &= \omega^2 \\ B_0 &= 3\omega^2 \end{aligned} \quad (2-14)$$

which is a straight line with slope of 3 passing through the origin.

The magnitude of the frequency response is defined by

$$A(\omega) = \frac{a_0}{|(2 - B_0) + j\omega(2 - B_1)|} = \frac{4}{|(4 - B_0) + j\omega(2 - B_1)|} \quad (2-15)$$

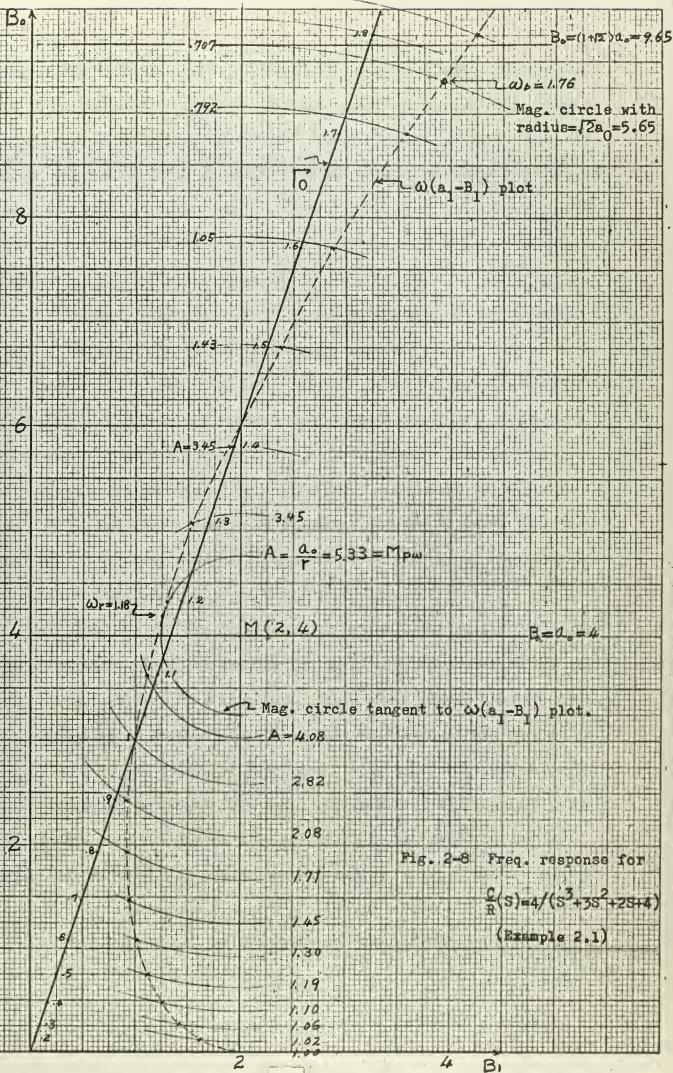
where  $B_1$  and  $B_0$  are defined by Equation (2-14).

The point M (2, 4) is located, and the lines  $B_1 = a_1 = 2$ ,  $B_0 = a_0 = 4$  and  $B_0 = (1 + \sqrt{2})a_0 = 9.65$  are drawn as shown. The  $\omega(a_1 - B_1)$  plot is constructed as illustrated in Section 2-1.

The  $\omega(a_1 - B_1)$  plot starts from the point (2, 0) and crosses the  $\Gamma_0$  curve at  $\omega = 1$  and the point (2, 6) at which the  $\Gamma_0$  curve and the line  $B_1 = 2$  intersect.

For  $0 < \omega < 1$  and for  $\omega > 1.42$  the  $\omega(a_1 - B_1)$  plot lies to the







right of the  $\Gamma_0$  curve and for  $1.0 < \omega < 1.42$  to the left. The point at  $\omega = .5$  on the  $\omega(a_1 - B_1)$  plot is located on a horizontal line passing through the point at  $\omega = .5$  on the  $\Gamma_0$  curve and the distance from the line  $B_1 = 2$  to the point at  $\omega = .5$  on the  $\omega(a_1 - B_1)$  plot is one-half of the distance from the line  $B_1 = 2$  to the point at  $\omega = .5$  on the  $\Gamma_0$  curve so that the distance from the  $B_1 = 2$  line to the point at  $\omega = .5$  on the  $\omega(a_1 - B_1)$  plot is equal to  $\omega(2 - B_1(\omega)) \Big|_{\omega = .5}$  which is the value of imaginary term at  $\omega = .5$  in the denominator of the equation defining the magnitude of the frequency response.

Some points on the  $\omega(a_1 - B_1)$  plot for some pertinent values of frequencies are determined likewise and those points are connected by a smooth curve to give the  $\omega(a_1 - B_1)$  plot as shown. Thus each pair of two points on a horizontal line, one on the  $\Gamma_0$  curve and the other on the  $\omega(a_1 - B_1)$  plot has one value of frequency and the distance from the point M (2, 4) to any point at frequency  $\omega$  on the  $\omega(a_1 - B_1)$  plot is the value of the denominator of Equation (2-15).

Magnitude circles are constructed as shown. Noting that the numerator of Equation (2-15) is constant at  $a_0 = 4$ , it is seen that the ratio of  $a_0 = 4$  to radius  $r$  of a magnitude circle which passes through a point at frequency of  $\omega$  on the  $\omega(a_1 - B_1)$  plot is the magnitude  $A(\omega)$ .

At  $\omega = 0$ , the radius of magnitude circle is 4 which gives  $A = 1$ . As the frequency increases from zero, the radius of the magnitude circle decreases, in turn increasing  $A$ . At  $\omega = 1.18$  magnitude circle is tangent to the  $\omega(a_1 - B_1)$  plot and has radius  $r = .75$  which implies

$$\text{resonance peak } M_{P\omega} = 4/.75 = 5.33$$

$$\text{at resonance frequency } \omega_r = 1.18$$



For frequencies from  $\omega = 1.18$  on, the radii of magnitude circles increase, in turn decreasing A.

A magnitude circle with radius of  $\sqrt{2} a_0 = 5.65$  crosses the  $\omega(a_1 - B_1)$  plot at  $\omega = 1.76$  which implies the bandwidth is

$$\omega_b = 1.76$$

The frequency response up to  $\omega = 1.8$  is thus represented on the  $B_0$  vs  $B_1$  plane.



## Example 2.2

Given an open loop transfer function of a unity feedback servo as

$$G(S) = \frac{121000(S+37)}{S(S+1.1)(S+1670)}$$

obtain the magnitude vs frequency plot of the frequency response.

The closed loop transfer function is

$$\frac{C}{R}(S) = \frac{121000S + 448000}{S^3 + 1671.1S^2 + 122671S + 448000} \quad (2-16)$$

Since Equation (2-16) deals with large numbers, it is not convenient to work on the  $B_0$  vs  $B_1$  plane associated with Equation (2-16).

By using a transformation or frequency scaling

$$S = 100s$$

Equation (2-16) transforms to

$$\frac{C}{R}(s) = \frac{12.1s + 4.48}{s^3 + 16.71s^2 + 12.267s + 4.48} \quad (2-17)$$

Then the equation defining frequency response is

$$\frac{C}{R}(j\omega_t) = \frac{|4.48 + j12.1\omega_t|}{|(4.48 - B_{0t}) + j\omega_t(12.27 - B_{1t})|} \quad (2-18)$$

where  $B_{1t}$  and  $B_{0t}$  are Mitrovic's variables of  $B_0$  vs  $B_1$  curve for  $\zeta = 0$  associated with the system (2-17) and  $\omega_t$  is related with  $\omega$ , the frequency associated with the system (2-16) by

$$\omega = 100 \omega_t$$

In Fig. 2-9 the  $\Gamma_0$  curve,  $\omega_t(a_{1t} - B_{1t})$  plot and magnitude circles representing the numerator of Equation (2-18) are constructed.

The frequency dependence of the imaginary term in the numerator of Equation (2-18) is represented on the  $B_{0t} = 2a_{0t}$  line to the left of



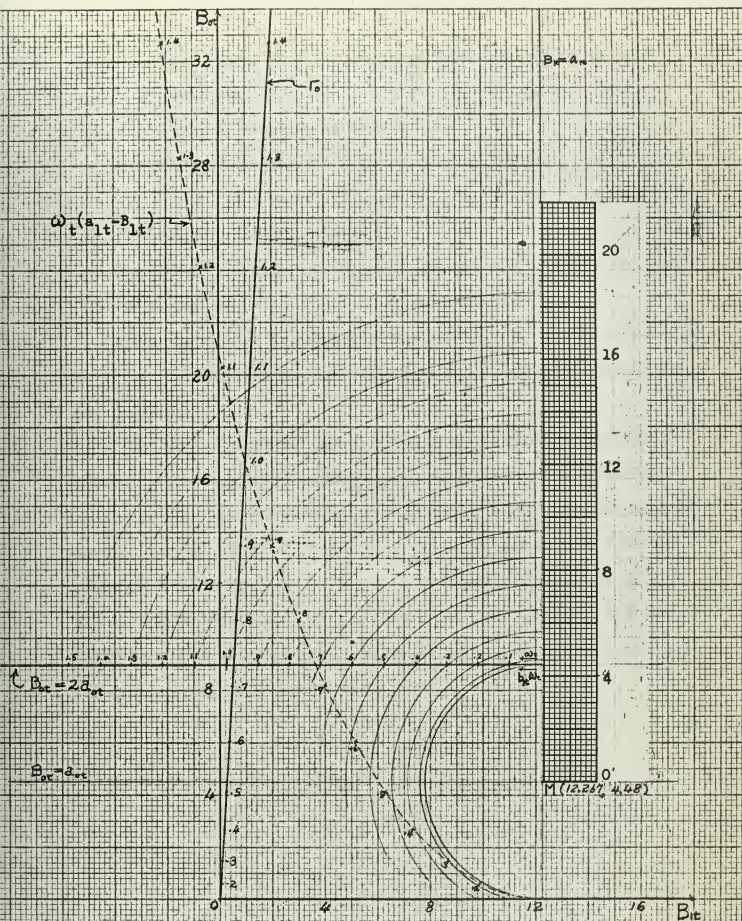


Fig. 2-9 Freq. response for  $C/R(s) = (12.1s + 4.48)/(s^3 + 16.71s^2 + 12.27s + 4.48)$   
(Example 2.2)



$B_{1t} = a_{1t}$  line. (Note that the frequency dependence of the imaginary term in the numerator of Equation (2-18) is represented on a line different from that illustrated in Section 2-1.)

As illustrated in Section 2-2, the distance from the point M (12.267, 4.48) to a point at frequency  $\omega_t$  on the  $B_{0t} = 2a_{0t} = 8.96$  line is equal to  $|4.48 + j12.267 \omega_t|$

The magnitude circles are drawn so that at any frequency the numerator and denominator of Equation (2-18) can be compared.

Since the M point is located at the point where the coordinates are not convenient numbers, reading off the radii of magnitude circles is not convenient.

This can be simplified by providing a scale which is calibrated with the same scale of coordinate axes so that the radius of any magnitude circle is read off easily. This is shown in Fig. 2-9, for example, at  $\omega_t = .5$ , the value of  $|N(j\omega_t)|$  at  $\omega_t = .5$  is read to be 7.5 on the scale provided on the line  $B_{1t} = 12.267$  by drawing a magnitude circle passing through the point at  $\omega_t = .5$  on the line  $B_{0t} = 2a_{0t}$  and similarly for the denominator  $|(a_{0t} - B_{0t}) + j\omega_t(a_{1t} - B_{1t})|$  at  $\omega_t = .5 = 6.0$ , thus the following table is constructed:



$\omega_c$	0	.2	.3	.4	.5	.6	.7	.8	.9	1.0	1.1	1.2	1.3	1.4
$ N(j\omega_c) $														
Numerator	4.48	5.1	5.7	6.5	7.6	8.5	9.5	10.6	11.7	12.8	14.0	15.2	16.3	17.4
$ F(j\omega_c) $														
Denominator	4.48	4.57	4.65	5.2	6.0	7.25	9.1	7.2	13.7	16.6	19.9	23.5	27.4	31.7
$A = \left  \frac{N}{F} (j\omega_c) \right $	1.0	1.11	1.225	1.25	1.25	1.17	1.04	.948	.854	.77	.705	.648	.595	.555

and the resulting magnitude vs frequency curve is shown in Fig. 2-10



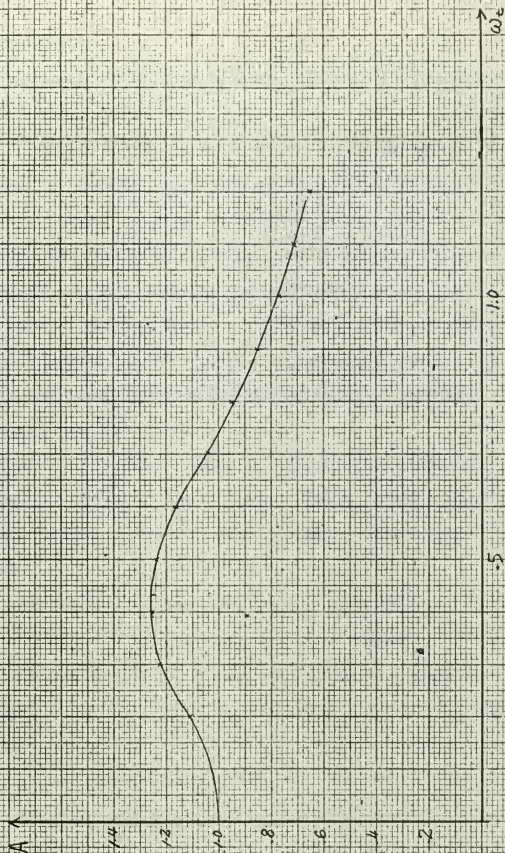


Fig. 2-10 Magnitude vs frequency plot.  
(Example 2.2)



### Example 2.3

Suppose for the system in Example 2.2 only the bandwidth is of interest. The equation

$$\left| 4.48 - B_{0t}(\omega_t) + j\omega_t (12.27 - B_{1t}(\omega_t)) \right| = \sqrt{2} \left| 4.48 + j12.1 \omega_t \right| \quad (2-19)$$

defines the bandwidth  $\omega_{bt}$  of the system in Equation (2-17).

By drawing a magnitude circle with radius equal to  $1.414(4.48 + j12.1)$  which is the quantity on the right side of the equation above at  $\omega_t = 1$ , and noting the frequencies in the neighborhood of the point on the  $\Gamma_0$  curve where the circle intersects with the  $\Gamma_0$  curve, and also noting the relative location of the  $\omega_t(a_{1t} - B_{1t})$  plot with respect to the  $\Gamma_0$  curve, some information can be obtained about the bandwidth. This is illustrated with Fig. 2-11.

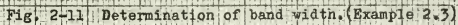
In Fig. 2-11, the  $\Gamma_0$  curve is drawn for the system (2-17), the working point M (12.27, 4.48) is located, and  $B_{1t} = a_{1t} = 12.27$  is laid off the same way as in Fig. 2-9. The line  $B_{0t} = (1 + 1.414)a_{0t} = 10.8$  is drawn.

From the intersection point of  $B_{0t} = 10.8$  and  $B_{1t} = 12.27$  lines to the left, a distance of  $1.414 b_{1t} = 17.1$  is measured off on the  $B_{0t} = 10.8$  line and labeled  $\omega_t = 1.0$ . The radius of a magnitude circle passing through the  $\omega_t = 1.0$  point on the  $B_{0t} = 10.8$  line is equal to  $|1.414(4.48 + j12.1)| = |1.414 N(j1)|$ . This circle denoted by  $C_{A1}$  crosses the  $\Gamma_0$  curve between the points at  $\omega_t = 1.0$  and  $\omega_t = 1.1$ .

It is easy to predict that  $\omega_{bt} < 1$  is not possible. If it is suspected that  $\omega_{bt} < 1$  is impossible, it can be checked as follows:

The check is based on the fact that, for the bandwidth, the magnitude circle must cross the  $\omega_t(a_{1t} - B_{0t})$  plot and  $1.414 b_{1t} \omega_t$  scale







represented on the line  $B_{0t} = (1 + \sqrt{2}) a_{0t} = 10.8$  at the same frequency, and the fact that if any magnitude circle is drawn and if it crosses the  $\omega_t(a_{1t} - B_{0t})$  plot and  $1.414 b_{1t} \omega_t$  scale ( $B_{0t} = 10.8$  line) at  $\omega_t = \omega_t''$  and  $\omega_t = \omega_t'$  respectively with either  $\omega' < \omega''$  or  $\omega' > \omega''$ , then for existence of a bandwidth frequency, there must be another magnitude circle to give a new set of  $\omega'$  and  $\omega''$  with inversed inequality from the previous set of  $\omega'$  and  $\omega''$ .

The magnitude circle  $C_{A1}$  gives

$$\omega_t'' > 1 \quad \text{and} \quad \omega_t' = 1 \quad \text{then} \quad \omega_t' < \omega_t''$$

Since  $C_{A1}$  is drawn with  $\omega_t' = 1$  and it crosses the  $\Gamma_0$  curve between  $\omega_t = 1.0$  and  $\omega_t = 1.1$  and  $1.0 < \omega_t'' < 1.1$ . This is reasoned without drawing any portion of the  $\omega_t(a_{1t} - B_{1t})$  plot.

Draw a magnitude circle with  $\omega_t' = 0$ . This circle is of radius equal to  $\sqrt{2} a_{0t}$  tangent to the line  $B_{0t} = (1 + \sqrt{2}) a_{0t} = 10.8$  and represents  $\sqrt{2} |N(j\omega)| = \sqrt{2} a_{0t}$  by its radius. Since  $\omega_t(a_{1t} - B_{1t})$  plot approaches to the point at  $\omega_t = 1$  on the  $\Gamma_0$  curve from the point (12.27, 0), the circle must cross the  $\omega_t(a_{1t} - B_{1t})$  plot at some frequency greater than zero to give an inequality

$$\omega_t' = 0, \quad \omega_t'' > 0 \quad \text{then} \quad \omega_t' < \omega_t''$$

The inequality sign is not reversed from the previous set of  $\omega_t'$  and  $\omega_t''$ , and this implies there might be no frequency such that, for  $0 < \omega_t < 1$ , a magnitude circle crosses the  $\omega_t(a_{1t} - B_{1t})$  plot and the  $\sqrt{2} b_{1t} \omega_t$  scale at the same frequencies; in other words,  $\omega_t' = \omega_t'' = \omega_{bt}$  may not be possible for the frequency range from zero to 1.0. However, there may be some frequency between 0 and 1.0 at which the inequality is reversed since the interval of  $\omega_t$  from 0 to 1.0 is equivalent to a relatively large interval of original system frequency  $\omega = 100 \omega_t$ .



Locate the point at  $\omega_t = .5$  on the  $\omega_t(a_{1t} - B_{1t})$  plot which is half way from the line  $B_{1t} = a_{1t} = 12.27$  to the point at  $\omega_t = .5$  on the  $\Gamma_0$  curve. This point is located inside the magnitude circle corresponding to  $\omega_t' = 0$ , therefore, no bandwidth is possible for  $\omega_t \leq .5$ .

For the magnitude circles corresponding to  $.7 < \omega_t' < 1$ , the inequality  $\omega' < \omega''$  is not changed. This can be checked by comparing the distances from the M point to the  $\Gamma_0$  curve and the  $\sqrt{2} b_{1t} \omega_t$  scale and noting that the  $\omega_t(a_{1t} - B_{1t})$  plot lies to the right of the  $\Gamma_0$  curve for  $.7 < \omega_t < 1.0$ .

For example, a magnitude circle for  $\omega_t' = .7$  will cross the  $\Gamma_0$  curve at a point where  $\omega_t$  is greater than .8 to give  $\omega_t'' > .8$  and  $\omega_t' < \omega_t''$ . Thus for the bandwidth,  $.7 < \omega_{bt} < 1.0$  is not possible.

The only part left to check is the interval  $.5 < \omega_t < .7$ . Knowing that

$$\begin{array}{lll} \text{for } \omega_t' = .5 & \omega' < \omega'' & \text{and} \\ \text{for } \omega_t' = .7 & \omega' < \omega'' & \end{array}$$

it is quite sure that for the interval  $.5 < \omega_t < .7$  the same inequality will be maintained. If a magnitude circle for  $\omega_t' = .6$  which is the mid point of the interval of the frequency from .5 to .7 is drawn, this circle will pass the  $\Gamma_0$  curve at the point where  $\omega_t = .6$  implying the point at  $\omega_t = .6$  on the  $\omega_t(a_{1t} - B_{1t})$  plot will be inside the circle in turn  $\omega_t' < \omega_t''$  which is the same inequality as the cases of  $\omega_t' = .5$  and  $\omega_t' = .7$ . Thus the same inequality is maintained for  $\omega_t$  from 0 to 1.0 and  $\omega_{bt} < 1$  is not possible.

The tests made above are carried out without plotting any portion of the  $\omega_t(a_{1t} - B_{1t})$  plot except the point at  $\omega_t = .5$  on the  $\omega_t(a_{1t} - B_{1t})$  plot is located. The tests can be carried out with a sketch of the



$\omega_t(a_{1t} - B_{1t})$  plot and the sketch can be made simply with reasonable accuracy by noting that the horizontal distance from the line  $B_{1t} = a_{1t} = 12.27$  to the  $\omega_t(a_{1t} - B_{1t})$  plot is a fraction of that to the  $\Gamma_0$  curve and the fraction is the value of  $\omega_t$ .

For an interval of frequencies ( $\omega_1, \omega_2$ ), if the inequality

$$\omega' < \omega'' \quad (\text{or } \omega' > \omega'')$$

holds at both ends  $\omega_1$  and  $\omega_2$  then the inequality is maintained throughout the interval unless the  $\Gamma_0$  curve or the  $\omega_t(a_{1t} - B_{1t})$  plot is oddly shaped in the interval, or in other words, unless the rates of change of the distance from the M point to the  $\omega_t(a_{1t} - B_{1t})$  plot is not fluctuating in the interval.

In this example, if the  $\omega_t(a_{1t} - B_{1t})$  plot is sketched, then one can see that the rate of change of distance from the M point to the  $\omega_t(a_{1t} - B_{1t})$  plot is steadily increasing except in the region of small frequencies.

Knowing that  $\omega_{bt} > 1.0$ , the next thing to do is to find another magnitude circle to insure an upper bound of  $\omega_{bt}$  as illustrated in Section 2-3.

Noting that  $C_{A1}$  crosses the  $\Gamma_0$  curve at the point where the frequency is slightly less than 1.1 and also noting that the  $\omega_t(a_{1t} - B_{1t})$  plot lies to the left of the  $\Gamma_0$  curve for  $\omega_t > 1$ , it is predicted that  $\omega_{bt}$  will not be very far from  $\omega_t = 1.1$ . Therefore it is natural to try a magnitude circle which passes through  $\omega_t = 1.1$  or  $\omega_t = 1.2$  on the  $\sqrt{2} b_{1t} \omega_t$  scale. The magnitude circle which passes the point at  $\omega_t = 1.2$  on the line  $B_{0t} = (1 + \sqrt{2}) a_{0t}$  is tried and denoted by  $C_{A2}$ . The circle  $C_{A2}$  represents the right side of Equation (2-19) at  $\omega_t = 1.2$ .



It passes the  $\sqrt{0}$  curve at a point where the frequency is less than 1.2 which implies it will pass a point on the  $\omega_t(a_{1t} - B_{1t})$  plot where frequency is less than 1.2, thus  $\omega_t = 1.2$  clearly indicates an upper bound of bandwidth and

$$1.0 < \omega_{bt} < 1.2$$

To determine  $\omega_{bt}$  precisely, the  $\omega_t(a_{1t} - B_{1t})$  plot for  $1.0 < \omega_t < 1.2$  is needed.

A magnitude circle  $C_{Ab}$  which passes through the point at  $\omega_t = 1.1$  on the  $\sqrt{2} b_{1t} \omega_t$  scale crosses the  $\omega_t(a_{1t} - B_{1t})$  plot at  $\omega_t = 1.1$  and bandwidth is determined to be

$$\omega_{bt} = 1.1$$

This agrees with Fig. 2-10.

Since Fig. 2-10 or 2.11 is for the system represented by Equation (2-17) which is related to the original system by  $S = 100s$ , the original system bandwidth  $\omega_b$  is

$$\omega_b = 100 \omega_{bt} = 110$$

By substituting  $s = j \omega_{bt} = 1.1$  and  $S = j \omega_b = 110$  into Equation (2-17) and (2-16) respectively, the magnitudes calculate to

$$\left| \frac{C}{R}(s) \right|_{s=j1.1} = .707$$

$$\left| \frac{C}{R}(S) \right|_{S=j110} = .707$$



2-4. Locus of M points on the  $B_0$  vs  $B_1$  plane for constant bandwidth.

In Section 2-3, bandwidth of a given system is determined. For a given system with fixed M point which is determined by the coefficients  $a_1$  and  $a_0$ , bandwidth is determined by varying  $\omega$  until some frequency (or frequencies)  $\omega = \omega_b$  is found to satisfy Equation (2-10).

Suppose, conversely,  $\omega$  in Equation (2-10) is fixed at bandwidth  $\omega_b$ , then there must be some  $a_1$  and  $a_0$  to satisfy Equation (2-10), and the pair  $(a_1, a_0)$  defines the M point on the  $B_0$  vs  $B_1$  plane for a system to have bandwidth equal to  $\omega_b$ . If all of such pairs  $(a_1, a_0)$  satisfying Equation (2-10) are plotted on the  $B_0$  vs  $B_1$  plane, then a curve which represents the locus of M points on the  $B_0$  vs  $B_1$  plane for fixed bandwidth  $\omega_b$  is obtained. This curve is called the constant bandwidth curve for bandwidth equal to  $\omega_b$  or merely constant  $\omega_b$  curve.

If a constant  $\omega_b$  curve is constructed on the  $B_0$  vs  $B_1$  plane, and if the M point is chosen at any point on the curve, then the resulting system will have bandwidth equal to  $\omega_b$ .

Since the synthesis of a feedback control system by Mitrovic's method is to choose a suitable M point to guarantee the required system performance specifications, the constant  $\omega_b$  curve will provide information in choosing the M point to meet specified bandwidth requirement.

By fixing  $\omega = \omega_b$  in Equation (2-10), the constant  $\omega_b$  curve is defined by

$$\left| (a_0 - B_0(\omega_b)) + j\omega_b(a_1 - B_1(\omega_b)) \right| = \sqrt{2} \left| N(j\omega_b) \right| \quad (2-20)$$

where  $B_1(\omega_b)$  and  $B_0(\omega_b)$  are the coordinates of the points on  $\Gamma_0$  curve at  $\omega = \omega_b$ .

On the left side of Equation (2-20),  $B_1(\omega_b)$  and  $B_0(\omega_b)$  are fixed and those values can be read off on the  $\Gamma_0$  curve.



On the right side of Equation (2-20), noting that  $N(S)$  is the numerator polynomial of the closed loop transfer function,  $|N(j\omega_b)|$  contains  $a_0$  which is treated as a variable on the constant  $\omega_b$  curve. Moreover, if  $N(S)$  is of order other than zero, it may contain  $b_1$  which is the coefficient of the  $S^1$  term in  $N(S)$ , and  $b_1$  is not independent of  $a_1$  which is the other variable on the constant  $\omega_b$  curve. Thus, even though Equation (2-20) is a perfectly general expression for defining constant  $\omega_b$  curves, it does not clearly indicate the geometrical shape of the constant  $\omega_b$  curve on the  $B_0$  vs  $B_1$  plane.

In the following, the cases of zero order, first order and second order of  $N(S)$  are considered.

#### $N(S)$ of zero order.

When a closed loop transfer function has no zeros,  $N(S) = a_0$  and Equation (2-20) becomes

$$\left| (a_0 - B_0(\omega_b)) + j\omega_b (a_1 - B_1(\omega_b)) \right| = \sqrt{2} a_0$$

by removing the absolute sign, and manipulating

$$(a_0 + B_0(\omega_b))^2 - \omega_b^2 (a_1 - B_1(\omega_b))^2 = 2 B_0^2(\omega_b) \quad (2-21)$$

Since  $\omega_b$ ,  $B_0(\omega_b)$ ,  $B_1(\omega_b)$  are fixed, Equation (2-21) indicates a hyperbola with following geometrical properties on the  $B_0$  vs  $B_1$  plane:

the center of symmetry at  $(B_1(\omega_b), -B_0(\omega_b))$

the vertices at  $(B_1(\omega_b), (-1 \pm \sqrt{2})B_0(\omega_b))$

major axis is  $B_1 = B_1(\omega_b)$

The two branches of hyperbola (2-21) lie symmetrically with respect to the  $B_0 = -B_0(\omega_b)$  line. The upper branch lies above the  $B_1$  axis with minimum point at  $(B_1(\omega_b), .414 B_0(\omega_b))$  which is one of the vertices.



For a stable system, only the portion of constant  $\omega_b$  curve which lies in the stable region is of interest.

It is interesting to note that the locus of the vertex of the branch of the hyperbola which lie in the first quadrant is similar to the  $\Gamma_o$  curve, and the locus of the center of the hyperbola is the mirror image of the  $\Gamma_o$  curve.

Those can be seen as follows. As shown previously the vertex of the upper branch of the hyperbola is at  $(B_1(\omega_b), .414B_0(\omega_b))$  and the center point is at  $(B_1(\omega_b), -B_0(\omega_b))$ .

Considering  $\omega_b$  as a continuous positive variable, then the locus of the vertex point is defined by

$$\begin{aligned} X &= B_1(\omega) \\ Y &= .414 B_0(\omega) \end{aligned}$$

and in like manner the locus of the center point is defined by

$$\begin{aligned} X &= B_1(\omega) \\ Y &= -B_0(\omega) \end{aligned}$$

where X and Y axes are identical to  $B_1$  and  $B_0$  axes respectively.

The two sets of equations can be compared with the  $\Gamma_o$  curve defined by

$$\begin{aligned} B_1 &= B_1(\omega) \\ B_0 &= B_0(\omega) \end{aligned}$$

Consider the consistency of principles involved in constant  $\omega_b$  curve and in determining bandwidth described in Section 2-3.

In Fig. 2-12, suppose a constant  $\omega_b$  curve is plotted from Equation (2-21) assuming the  $\Gamma_o$  curve as given. The point P is at  $\omega = \omega_b$  on the  $\Gamma_o$  curve. The constant  $\omega_b$  curve is denoted by  $C_{\omega_b}$ . The vertical axis denoted by Y is on the  $B_1 = B_1(\omega_b)$  line which passes through the



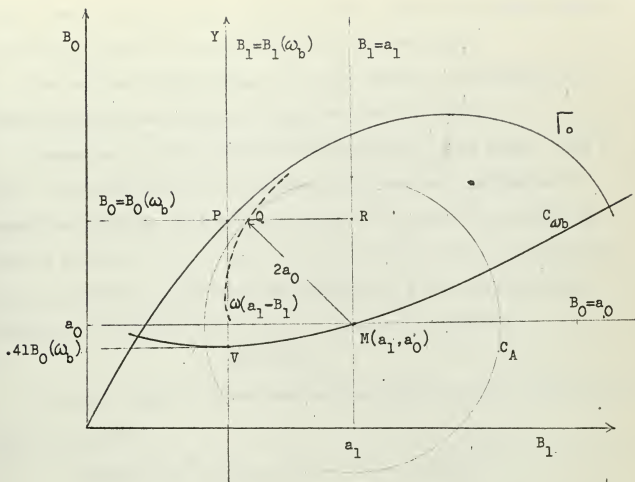


Fig. 2-12 Showing consistency of Section 2-3 and  
Section 2-4 about bandwidth.



point P and V is the vertex. The horizontal axis and the lower branch of the hyperbola which lie below the  $B_1$  axis are not shown.

The portion of the constant  $\omega_b$  curve which is of interest for a stable system is the portion of  $C_{\omega_b}$  under the  $\Gamma_0$  curve.

Suppose an M point is chosen arbitrarily on  $C_{\omega_b}$  as shown. Then a  $\omega(a_1 - B_1)$  plot results as described in the previous sections and a magnitude circle  $C_A$  with radius equal to  $\sqrt{2} a_0$  must pass through the point Q at which  $\omega = \omega_b$  on the  $\omega(a_1 - B_1)$  plot to have a consistency for the principles involved in the constant  $\omega_b$  curve and determining bandwidth as described in Section 2-3. This is to show that

$$\overline{MQ} = \sqrt{2} a_0$$

Locating a point R at the intersection point of the  $B_1 = a_1$  line and  $B_0 = B_0(\omega_b)$  line,

$$\overline{MQ} = (\overline{MR}^2 + \overline{RQ}^2)^{1/2} = [(B_0(\omega_b) - a_0)^2 + (\omega_b(a_1 - B_1(\omega_b)))^2]^{1/2}$$

But  $(a_1, a_0)$  satisfies Equation (2-21) and it rearranges to

$$(a_0 - B_0(\omega_b))^2 + \omega_b^2 (a_1 - B_1(\omega_b))^2 = 2 a_0^2$$

and  $\overline{MQ} = \sqrt{2} a_0$

#### N(S) of first order.

When a closed loop system has one zero

$$N(S) = b_1 S + a_0$$

For unity feedback control system,  $b_1$  enters into  $a_1$  and  $a_1$  and  $b_1$  are not independent of each other.

Suppose an open loop transfer function of a unity feedback control system as

$$G(S) = \frac{b_1 S + a_0}{S^n + d_{n-1} S^{n-1} + \dots + d_1 S}$$



where  $b_1$ ,  $a_1$  and  $a_0$  are variable and  $d_k$ 's are fixed constants.

$$\text{Then } \frac{C}{R}(s) = \frac{b_1 s + a_0}{s^n + d_{n-1} s^{n-1} + \dots + (d_1 + b_1) s + a_0}$$

$$\text{or } \frac{C}{R}(s) = \frac{b_1 s + a_0}{s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0} \quad (2-22)$$

where  $a_{n-1} = d_{n-1}$ ,  $\dots$ ,  $a_2 = d_2$  and  $a_1 = d_1 + b_1$

$N(s)$  can be written in the form of

$$N(s) = (a_1 - d_1) s + a_0$$

With this  $N(s)$ , Equation (2-20) is manipulated to

$$[a_0 + B_0(\omega_b)]^2 + \omega_b^2 [a_1 + (B_1(\omega_b) - d_1)]^2 = D(\omega_b, d_1) \quad (2-23)$$

where  $D(\omega_b, d_1) = 2[B_0^2(\omega_b) + \omega_b^2 (B_1(\omega_b) - d_1)^2] + d_1^2 \omega_b^2$

Equation (2-23) defines the constant  $\omega_b$  curve for any system whose closed loop transfer function has one zero.

Noting that for  $D(\omega_b, d_1) > 0$ , Equation (2-23) is an ellipse centered at  $(2d_1 - B_1(\omega_b), -B_0(\omega_b))$  with semi-axes of lengths equal to

$$\frac{\sqrt{D(\omega_b, d_1)}}{\omega_b} \quad \text{and} \quad \sqrt{D(\omega_b, d_1)} \quad . \quad (\text{Horizontal and vertical axes respectively.})$$

For each  $d_1$ , as  $\omega_b$  varies there results a family of ellipses.

It is seen in Section 2-3 that for a system represented by Equation (2-22), larger  $b_1$  results in larger bandwidth.

For any unity feedback system assuming all positive coefficients in its open loop transfer function,  $b_1$  is always less than  $a_1$  and its upper limit is  $a_1$  which is the case of a type 2 system, whereas its lower limit is zero which is the case of a system with no zeros.

For the upper limit of  $b_1 = a_1$  corresponds to  $d_1 = 0$ , and letting



$d_1 = 0$  in Equation (2-23), the constant  $\omega_b$  curve is then

$$(\bar{a}_0 + B_0(\omega_b))^2 + \omega_b^2 (\bar{a}_1 + B_1(\omega_b))^2 = 2(B_0^2(\omega_b) + \omega_b^2 B_1(\omega_b)) \quad (2-24)$$

Equation (2-24) can be applied for a type 2 system with one zero in its closed loop transfer function. For any system with  $b_1 < a_1$  or  $d_1 > 0$ , if the constant  $\omega_b$  curve is plotted by Equation (2-24), and if the M point is located on the curve, then the system will have bandwidth less than  $\omega_b$ , thus Equation (2-24) defines a locus of M points on the  $B_0$  vs  $B_1$  plane for some upper bound of bandwidth for a system whose closed loop transfer function is represented in the form of Equation (2-22).

This will be illustrated with a specific system

$$\frac{C}{R}(S) = \frac{b_1 S + a_0}{S^4 + S^3 + S^2 + a_1 S + a_0} \quad (2-25)$$

with associated constant  $\omega_b$  curves as shown in Fig. 2-13.

For the system (2-25) the  $\Gamma_0$  curve is defined by

$$\begin{aligned} B_1 &= \omega^2 \\ B_0 &= \omega^2 - \omega^4 \end{aligned}$$

and this curve is plotted in Fig. 2-13 as shown.

If the system had no zero, then constant  $\omega_b$  curves would be defined by Equation (2-21) with  $B_1$  and  $B_0$  defined as above. For this case, constant  $\omega_b$  curves are constructed for  $\omega_b = .1$  to  $\omega_b = .5$  as shown in Fig. 2-13. If M point is chosen at any point on  $C_{.5}$  which is a constant  $\omega_b$  curve for  $\omega_b = .5$  then the system

$$\frac{C}{R}(S) = \frac{a_0}{S^4 + S^3 + S^2 + a_1 S + a_0} \quad (2-26)$$

will have bandwidth equal to .5.



The dotted curve which is a parabola is the locus of the vertices of the constant  $\omega_b$  curves for the system (2-26).

A constant  $\omega_b$  curve of  $\omega_b = 1.0$  for a system

$$\frac{C}{R}(S) = \frac{a_1 S + \bar{a}_0}{S^4 + S^3 + S^2 + a_1 S + \bar{a}_0} \quad (2-27)$$

which is obtained by letting  $b_1 = a_1$  in Equation (2-25) is plotted by Equation (2-24) and denoted by  $C_{1.0}$  in Fig. 2-13. If the axes scales are made the same, the  $C_{1.0}$  would be an arc of a circle, since, for  $\omega_b = 1$ ,  $B_0(1) = 0$ ,  $B_1(1) = 1$  then Equation (2-24) is

$$\bar{a}_0^2 + (\bar{a}_1 + 1)^2 = 2$$

which is a circle centered at  $(-1, 0)$  with radius equal to  $\sqrt{2}$ .

If the M point is chosen at any point on  $C_{1.0}$  then the system (2-27) will have bandwidth equal to 1.0.

Suppose the point M(.407, .089) is chosen for both systems (2-26) and (2-27). Since the point M(.407, .089) is on both  $C_{.5}$  and  $C_{1.0}$ , the systems (2-26) and (2-27) have bandwidth  $\omega_b = .5$  and  $\omega_b = 1.0$  respectively.

For system (2-26)

$$\left| \frac{C}{R}(j.5) \right| = \left| \frac{.089}{S^4 + S^3 + S^2 + .407S + .089} \right|_{S=j.5} = .707$$

For system (2-27)

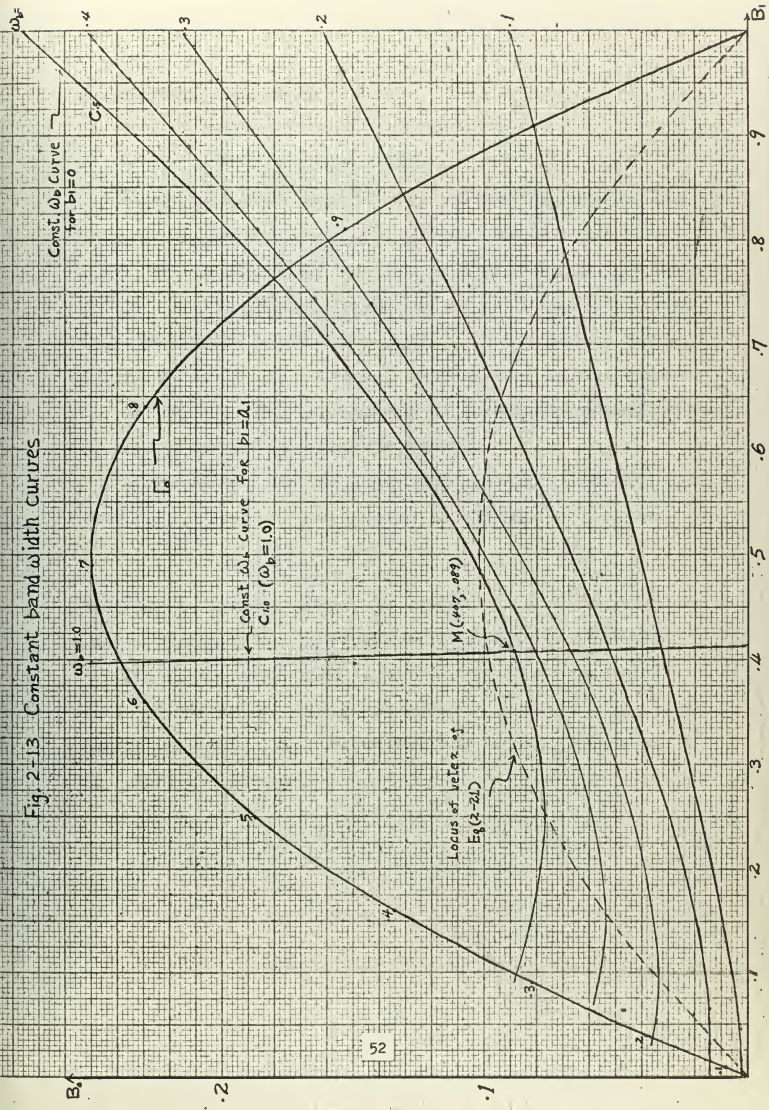
$$\left| \frac{C}{R}(j1.0) \right| = \left| \frac{.407S + .089}{S^4 + S^3 + S^2 + .407S + .089} \right|_{S=j1.0} = .707$$

Then the point M(.407, .089) for the system (2-25) guarantees that

$$\begin{aligned} 0.5 &< \omega_b < 1.0 \\ \text{for } 0 &< b_1 < a_1 \end{aligned}$$



Fig. 2-13 Constant band width curves





### N(S) of second order and up.

When a closed loop system has 2 or more zeros

$$N(S) = a_0 + b_1 S + b_2 S^2 + \dots$$

If the coefficients  $b'_k S$  for  $k \geq 2$  are fixed (in order for a system to be analyzed or synthesized by Mitrovic's method on the  $B_0$  vs  $B_1$  plane, the coefficients  $a'_k S$  for  $k \geq 2$  must be determined and likewise for  $b'_k S$ ), then by proceeding in the same way as the case of  $N(S) = b_1 S + a_0$ , the equation defining the constant  $\omega_b$  curve, which is similar to Equation (2-23) with due modification to account for the coefficients  $b_2, b_3, \dots$  will be obtained and likewise for the equation defining constant  $\omega_b$  curve of certain upper bound for bandwidth.

It must be noted that for a high-velocity constant system  $b_1$  is very close to  $a_1$  since the velocity constant  $K_v$  is related to  $a_1, a_0$  and  $b_1$  by

$$K_v = \frac{a_0}{a_1 - b_1}$$

Therefore, for a high  $K_v$  system, the constant  $\omega_b$  curve obtained by letting  $b_1 = a_1$  will be a good approximation to the actual bandwidth.

Also in lag compensation,  $a_1 \approx b_1$  and as shown in references 1 and 2, lag compensation problems are solved by letting  $a_1 = b_1$ , consequently, the constant  $\omega_b$  curve obtained from  $a_1 = b_1$  will be good in lag compensation problems.



## 2-5. Bandwidth of third order systems.

It is shown in references 1 and 2 that the equations of third order systems can be normalized to give a universal Mitrovic's chart for third order systems, and this chart is constructed in references 1 and 2.

In the normalized chart,  $\Gamma'_0$  is a straight line extending from the origin to infinity with slope of unity on the  $B_{0n}$  vs  $B_{1n}$  plane, and the frequency  $\omega_{tn}$  in the normalized system is related to the frequency  $\omega$  in the unnormalized system by

$$\omega = k \omega_{tn}$$

where  $k$  is a constant, the normalizing factor, and subscript  $n$  denote normalized quantity.

Assume a third order system with no zeros in its closed loop transfer function:

$$\frac{C}{R}(S) = \frac{a_0}{S^3 + a_2 S^2 + a_1 S + a_0}$$

It normalizes to, by transformation  $S = a_2 s$

$$\frac{C}{R}(s) = \frac{a_{0n}}{s^3 + s^2 + a_{1n}s + a_{0n}} \quad (2-28)$$

$$\begin{aligned} \text{where } a_{1n} &= a_1 / a_2^2 \\ a_{0n} &= a_0 / a_2^3 \end{aligned} \quad (2-29)$$

Equation (2-29) defines the relation between coefficients of the normalized and unnormalized systems, consequently the Mitrovic's variables are related by Equation (2-29) between normalized and unnormalized systems,

$$\begin{aligned} \text{namely } B_{1n} &= B_1 / a_2^2 \\ B_{0n} &= B_0 / a_2^3 \end{aligned} \quad (2-30)$$



Applying Equation (2-21) to the system represented by Equation (2-28), the constant  $\omega_b$  curve is defined by

$$(a_{0n} + B_{0n}(\omega_{bn}))^2 + \omega_{bn}^2 (a_{1n} - B_{1n}(\omega_{bn}))^2 = 2 B_{0n}^2(\omega_{bn}) \quad (2-31)$$

where  $\omega_{bn}$  is the bandwidth of normalized system and is related to the unnormalized bandwidth  $\omega_b$  by  $\omega_b = a_2 \omega_{bn}$ .

$B_{1n}$  and  $B_{0n}$  are Mitrovic's variables for the  $\Gamma_0$  curve in the normalized third order chart, and are related with unnormalized variables  $B_1$  and  $B_0$  by Equation (2-30).

Equation (2-31) defines a family of constant  $\omega_{bn}$  curves on the  $B_{0n}$  vs  $B_{1n}$  plane if a set of  $\omega_{bn}$  is supplied.

If this family of constant  $\omega_{bn}$  curves are superposed on a third order chart, then it will provide additional information about the third order system.

Fig. 2-14 shows the family of constant bandwidth curves for third order systems whose closed loop transfer function is represented by Equation (2-28) on the normalized  $B_0$  vs  $B_1$  plane. Note that the portions of constant bandwidth curves defined by Equation (2-31) which lie in the unstable region are omitted.

Fig. 2-14, the constant bandwidth chart for third order systems, can be superposed on Fig. 2-15 which is the third order Mitrovic's chart constructed in references 1 and 2 to provide additional information concerning system bandwidth in both analysis and synthesis of third order systems.

To determine system bandwidth is to locate the M point of given system on the constant bandwidth chart and consult with the constant



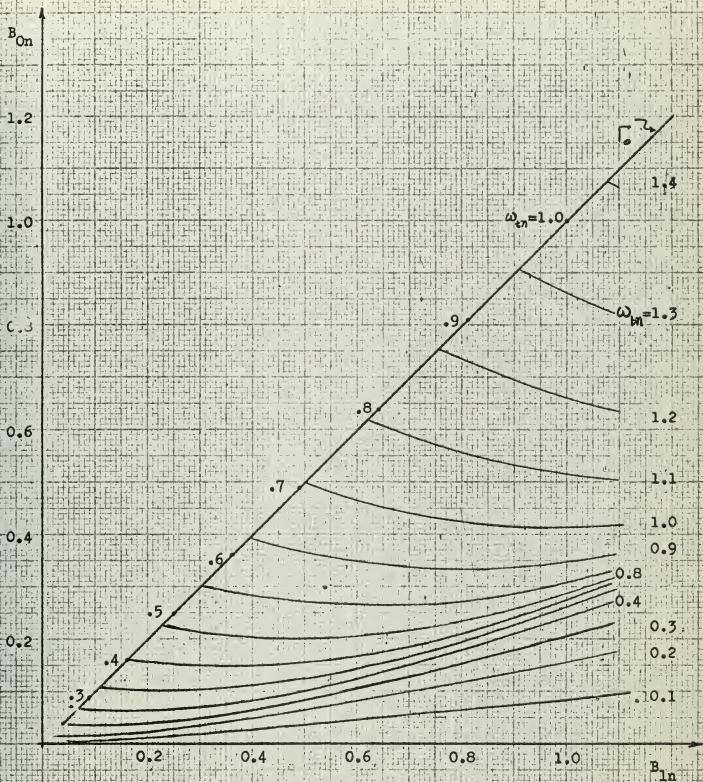


Fig. 2-14 Constant bandwidth curves

(3rd order systems-normalized, for  $b_{ln}=0$ )



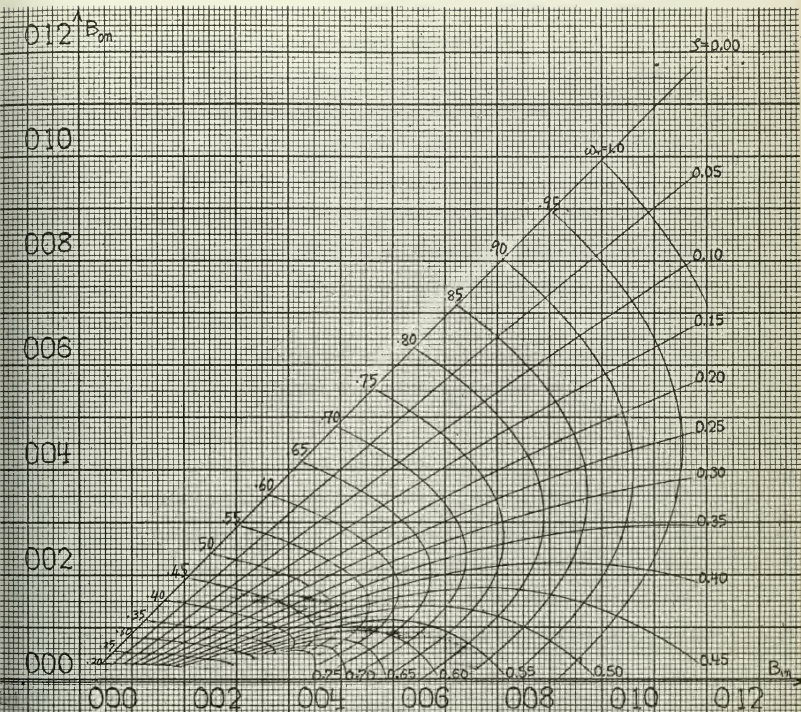


Fig 2-15 3rd order chart(Normalized  $B_0$  vs  $B_1$  curves)

AXIS SCALE =  $2.00E-01$   
 AXIS SCALE =  $2.00E-01$



bandwidth curves. Also it is an easy matter to choose an M point such that it guarantees a given bandwidth specification.

From Fig. 2-14, the effect of moving the M point (in the stable region) on system bandwidth can be analyzed as follows:

Suppose a point M ( $a_{1n}, a_{0n}$ ) is located in the stable region. For fixed  $a_{1n}$ , increasing  $a_{0n}$  results in increasing the bandwidth as is easily seen from Fig. 2-14. This agrees with common experience.

For fixed  $a_{0n}$ , the effect of changing  $a_{1n}$  on bandwidth is dependent upon the value of  $a_{0n}$ . This can be analyzed by noting the slope of constant  $\omega_{bn}$  curves.

It is seen in Section 2-4 that the locus of the vertex point of the family of constant bandwidth curves, the hyperbolas, is

$$X = B_{1n}(\omega_{tn})$$

$$Y = 0.414 B_{0n}(\omega_{tn})$$

where X and Y axes are identical to the  $B_{1n}$  and  $B_{0n}$  axes respectively and  $B_{1n}(\omega_{tn})$  and  $B_{0n}(\omega_{tn})$  are the equations which define the  $\Gamma_0$  curve. Noting that the  $\Gamma_0$  curve for the normalized system is a straight line emanating from the origin with unity slope in the first quadrant, then the locus of the vertex points of the constant  $\omega_{bn}$  curves is a straight line with slope of .414 defined by

$$B_{0n} = .414 B_{1n} \quad (2-32)$$

on the  $B_{0n}$  vs  $B_{1n}$  plane.

Since a vertex point of a hyperbola is the point at which the slope of the hyperbola changes its sign, every constant  $\omega_{bn}$  curve lying in the stable region is such that the portion of the curve which is above (to the left of) the line defined by Equation (2-32) has negative slope and dually.



Dividing the stable region into two regions by the line defined by Equation (2-32), it is easy to see that if  $a_{0n}$  is fixed, then by increasing  $a_{1n}$ , the bandwidth is increased as far as the M point is in the region above the line and dually for the region below the line.

If a closed loop system has a zero in its closed loop transfer function, then the normalized system equation is

$$\frac{C}{R}(s) = \frac{b_{1n}s + a_{0n}}{s^3 + s^2 + a_{1n}s + a_{0n}} \quad (2-33)$$

where  $b_{1n}$  is normalized  $b_1$ , the coefficient of  $s^1$  term in the numerator polynomial of the unnormalized system, and the relation between  $b_1$  and  $b_{1n}$  is

$$b_{1n} = b_1 / a_2^2$$

The arguments developed in Section 2-4 for a system whose closed loop transfer function has one zero applies to the third order system represented by Equation (2-33).

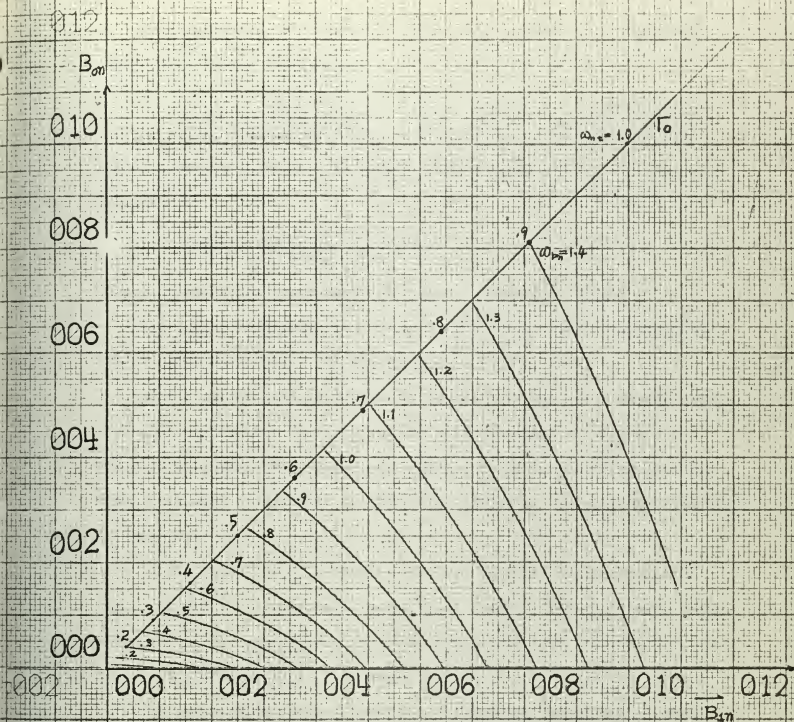
By letting  $b_{1n} = a_{1n}$  in Equation (2-33), then the resulting constant bandwidth curves which are defined by Equation (2-24) in Section 2-4 (with due account of normalized third order  $\Gamma_0$ ) are:

$$(\bar{a}_{0n} + B_{1n}(\omega_{bn}))^2 + \omega_{bn}^2 (\bar{a}_{1n} + B_{1n}(\omega_{bn}))^2 = 2 [B_{0n}^2(\omega_{bn}) + \omega_{bn}^2 B_{1n}^2(\omega_{bn})] \quad (2-34)$$

where  $B_{1n}$  and  $B_{0n}$  are Mitrovic's variables for the  $\Gamma_0$  curve on the normalized third order charts.

The portions of the ellipses lying in the stable region are plotted in Fig. 2-16 which can also be superposed on Fig. 2-15 to provide information about bandwidth for any third order system with one zero in its closed loop system transfer function.





X AXIS SCALE =  $2.00E-01$   
 Y AXIS SCALE =  $2.00E-01$

Fig 2-16 Constant bandwidth curves

(Normalized 3rd order system

for  $b_{1n} = a_{1n}$ )



As illustrated in Section 2-4, Fig. 2-16 constitutes a bandwidth chart for a system represented by

$$\frac{C}{R}(S) = \frac{a_1 S + a_0}{S^3 + a_2 S^2 + a_1 S + a_0} \quad \text{or} \quad \frac{C}{R}(s) = \frac{a_{1n} + a_{0n}}{s^3 + s^2 + a_{1n} s + a_{0n}}$$

and a bandwidth chart for the upper bound of bandwidth for a system represented by

$$\frac{C}{R}(S) = \frac{b_1 S + a_0}{S^3 + a_2 S^2 + a_1 S + a_0}$$

or equivalently for the normalized system in Equation (2-33).

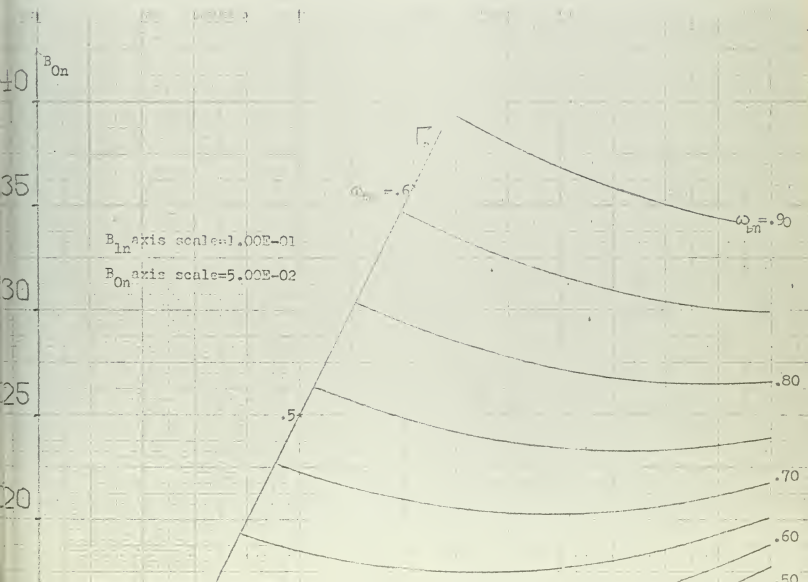
The effect of moving the M point in the stable region on bandwidth can be analyzed in Fig. 2-16 analogously as in Fig. 2-14.

Note that the ellipse defined by Equation (2-34) has a center point moving along the line  $B_{0n} = B_{1n}$  in the third quadrant as  $\omega_{bn}$  varies, and also note that the ellipse has negative slope in the first quadrant. Thus it is easy to see that by increasing either  $a_{1n}$  or  $a_{0n}$ , the bandwidth is increased.

In Fig. 2-17, 2-18, 2-20 and 2-21, the constant bandwidth charts, Fig. 2-14 and 2-16 are plotted on expanded scales. The third order chart Fig. 2-15 is correspondingly expanded in Fig. 2-19 and 2-22 which go with Fig. 2-17, 2-18 and 2-20, 2-21 respectively.

The following examples illustrate determining bandwidths of third order closed loop systems by using the bandwidths charts developed in this section, and also illustrate a design of system to meet exact bandwidth by using the third order system bandwidth chart.







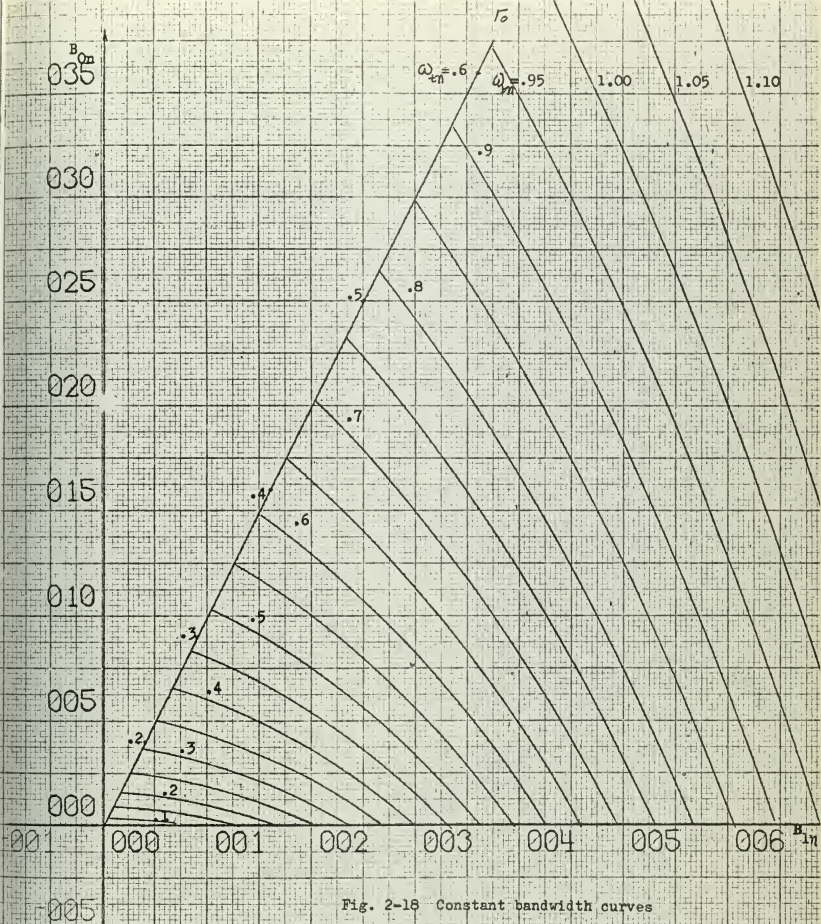


Fig. 2-18 Constant bandwidth curves

(Normalized 3rd order system,  $b_{1n} = a_{1n}$ )

X AXIS SCALE =  $1.00E-01$

Y AXIS SCALE =  $5.00E-02$

CHOICE 1101 YON 3RD ORDER B

CONSTANT BANDWIDTH



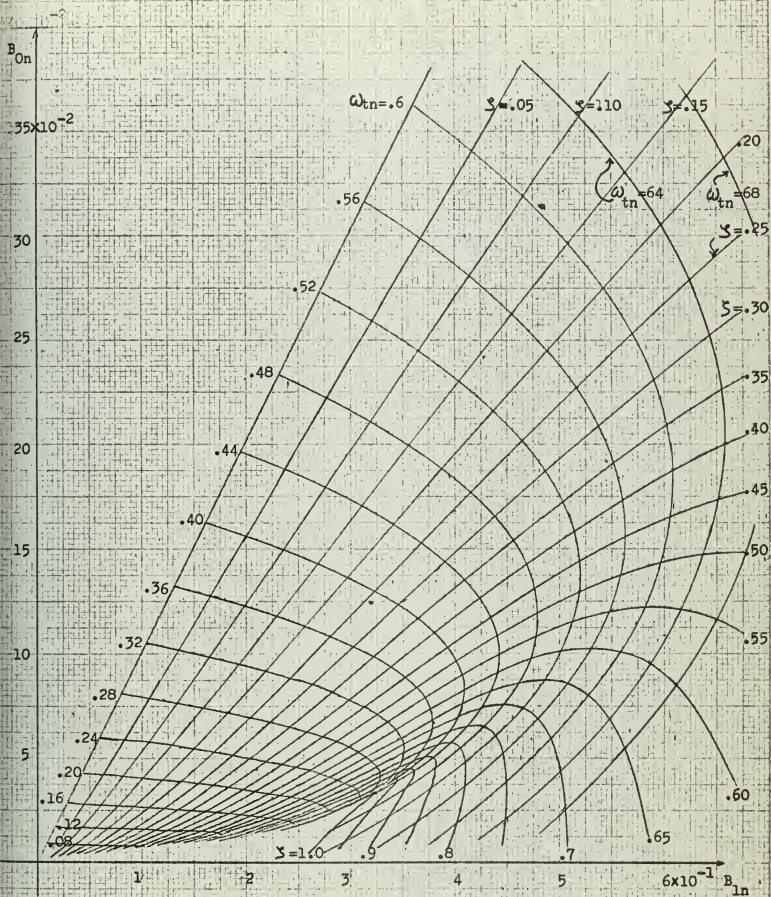


Fig. 2-19 3rd order chart (Normalized  $B_0$  vs  $B_1$  curves)



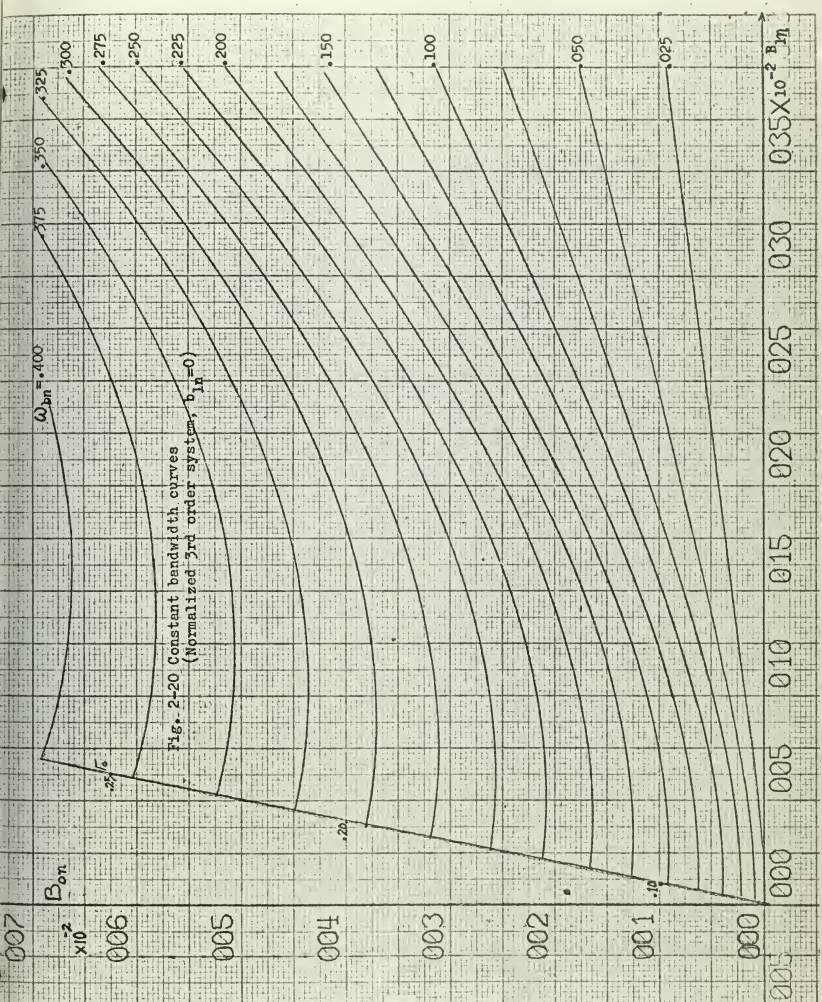




Fig. 2-21. Constant bandwidth curves

(Normalized 3rd order system,  $b_{1n}=a_{1n}$ )

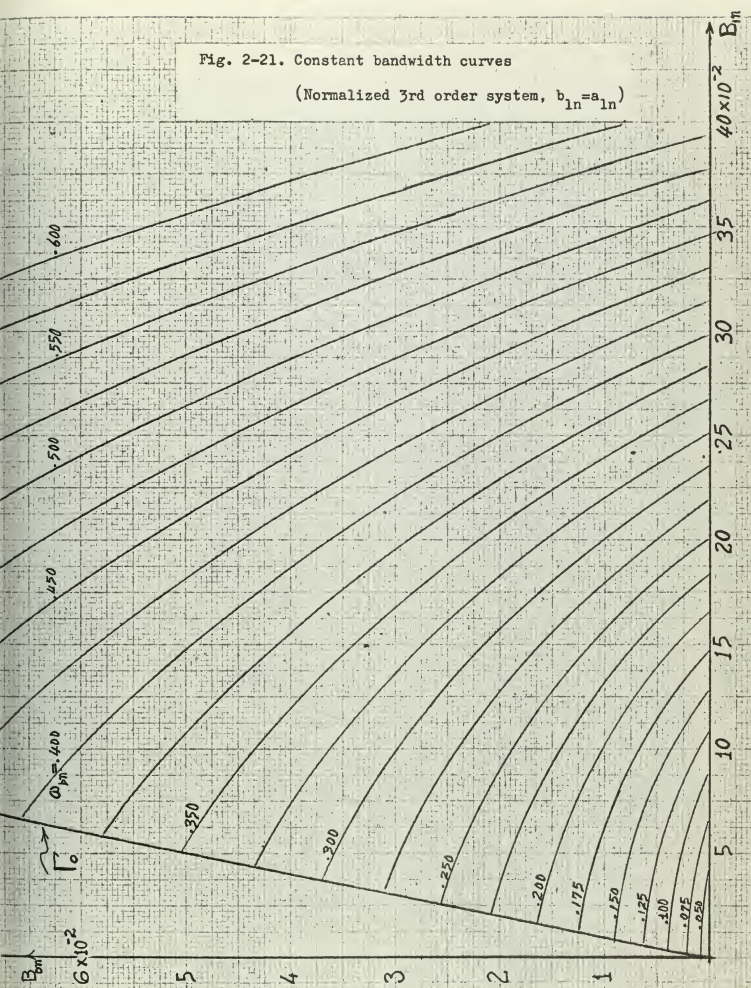
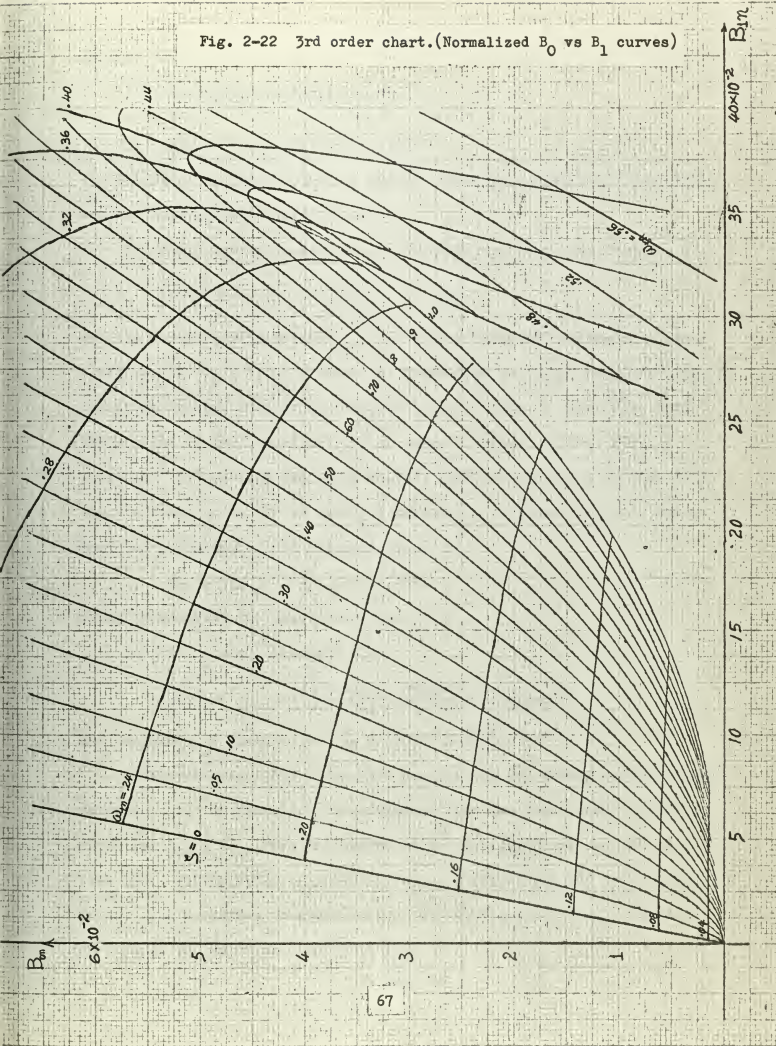




Fig. 2-22 3rd order chart.(Normalized  $B_0$  vs  $B_1$  curves)





#### Example 2.4

For the system in Example 2.1

$$\frac{C}{R}(S) = \frac{4}{S^3 + 3S^2 + 2S + 4}$$

the bandwidth is found to be  $\omega_b = 1.76$ . Now one can find the bandwidth on the third order bandwidth chart.

The given system normalizes to, by using the transformation  $S = 3s$ ,

$$\frac{C}{R}(s) = \frac{.148}{s^3 + s^2 + .222s + .148}$$

and the normalized bandwidth  $\omega_{bn}$  is related with the unnormalized bandwidth  $\omega_b$  by  $\omega_b = 3\omega_{bn}$ . Since the system has no zeros, one enters into the chart in Fig. 2-14 or the expanded chart in Fig. 2-17 or Fig. 2-20.

The M point of the normalized system seems the chart in Fig. 2-17 is suitable. By locating the point M(.222, .148) on the chart in Fig. 2-17,  $\omega_{bn}$  is found to be .59 by interpolating  $\omega_{bn} = .55$  and  $\omega_{bn} = .6$  curves. Then the bandwidth of the original system is

$$\omega_b = 3\omega_{bn} = (3)(.59) = 1.77$$

which agrees with the result in Example 2.1.

For the system in Example 2.2

$$\frac{C}{R}(S) = \frac{121000S + 448000}{S^3 + 1671.1S^2 + 122671S + 448000}$$

the bandwidth is found to be  $\omega_b = 110$  in Example 2.3.

This system is third order with one zero in its closed loop transfer function. If the bandwidth is determined from the third order bandwidth chart for  $a_{1n} = b_{1n}$  then the bandwidth so determined must be not smaller than 110. Noting that  $a_1 = 122671$  and  $b_1 = 121000$  are not greatly different, the bandwidth determined from the chart for  $b_{1n} = a_{1n}$  will be very close to the actual bandwidth 110.



The system normalizes to, through the transformation  $S = 1671.1s$ ,

$$\frac{C}{R}(s) = \frac{b_n s + .00096}{s^3 + s^2 + .0438s + .00096}$$

Entering into the bandwidth chart in Fig. 2-21 with the normalized  $M(.0438, .00096)$  point, the normalized bandwidth is found to be

$$\omega_{bn} = .07$$

by interpolating between the two curves for  $b_n = .05$  and  $b_n = .075$ . Then the unnormalized bandwidth is

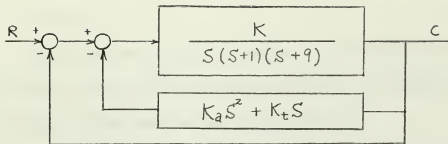
$$\omega_b = a_s \omega_{bn} = (1671.1)(.07) = 117$$

which is very close to the actual bandwidth  $\omega_b = 110$  and agrees with the arguments developed in Sections 2-3, 2-4 and 2-6.



### Example 2.5

Given a system block diagram below



adjust  $K, K_a$  and  $K_t$  with the following performance specifications:

- 1)  $\zeta \geq .3$
- 2)  $K_v = 10$
- 3)  $\omega_b = 20$

For the system without feeding back derivative signals  $K$  must be set to 90 in order to meet the  $K_v$  specification, and the system is at stability limit which can be seen from the third order chart.

The closed loop transfer function with the derivative signals feeded back is

$$\frac{C}{R}(S) = \frac{K}{S^3 + (10 + KK_a)S^2 + (9 + KK_t)S + K}$$

and normalizes to, by using the transformation  $S = a_3 s = (10 + KK_a)s$ ,

$$\frac{C}{R}(s) = \frac{a_{0n}}{s^3 + s^2 + a_{1n}s + a_{0n}}$$

$$\text{where; } a_{1n} = (9 + KK_t) / (10 + KK_a)^2$$

$$a_{0n} = K / (10 + KK_a)^3$$

$$\text{and } a_3 = 10 + KK_a = \text{normalizing factor.}$$

The velocity constant and bandwidth of the unnormalized and normalized systems are related by



$$K_v = (10 + K_a) K_{vn}$$

$$\omega_b = (10 + K_a) \omega_{bn}$$

$$K_v / \omega_b = K_{vn} / \omega_{bn} = 1/2$$

where the subscript "n" denotes the normalized system quantity.

Thus the ratio of the velocity constant  $K_{vn}$  and the bandwidth  $\omega_{bn}$  of the normalized system must be maintained at 1/2.

In Fig. 2-22a the  $\Gamma_0$  and  $\Gamma_{.3}$  curves are duplicated from the third order chart (Fig. 2-15) and then some constant bandwidth curves are superposed from the third order system bandwidth chart (Fig. 2-14).

In order for the system to have  $\zeta \geq .3$  the M point must be located below the  $\Gamma_{.3}$  curve in the first quadrant on the  $B_{0n}$  vs  $B_{1n}$  plane. For the normalized system  $K_{vn} = a_{0n}/a_{1n}$  indicates that for an M point located anywhere on the normalized  $B_{0n}$  vs  $B_{1n}$  plane if a line is drawn from the origin to the M point, then the slope of the line is equal to the normalized velocity constant  $K_{vn}$ . Therefore for this problem the M point must be at the intersection point of a constant  $\omega_{bn} = \omega'_{bn}$  curve and a line drawn from the origin such that the slope of the line is  $(1/2)\omega'_{bn}$  and moreover that M point must be such that it insures  $\zeta \geq .3$ . For any chosen constant  $\omega_{bn} = \omega'_{bn}$  curve the corresponding straight line with slope  $1/2 \omega'_{bn}$  can be drawn and then the line almost always intersects with the constant  $\omega_{bn} = \omega'_{bn}$  curve. When intersection occurs, location of the M point at the intersection guarantees that both  $K_v$ , bandwidth specifications have been satisfied, but may not guarantee the  $\zeta$  specification. In order that all three specifications be satisfied, the intersection must lie in the area enclosed by the  $\Gamma_{.3}$  curve and the  $B_{1n}$  axis in the first quadrant.



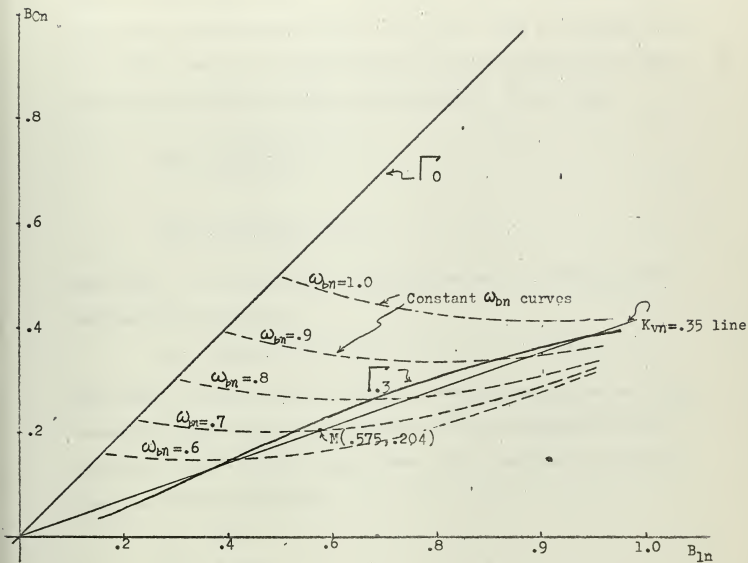


Fig. 2-22a Solution of Example 2.5 using  
the 3rd order system band width chart.



If one chooses the constant bandwidth curve for  $\omega_{bn} = 0.7$  and draws the line with a slope of 0.35 from the origin as shown in Fig. 2-22a, then the intersection point is at (.575, .204) and this point insures  $\Sigma \geq .3$  which is evident from the  $\Gamma_3$  curve configuration, thus one chooses the M point at  $M(a_{1n}, a_{0n}) = M(.575, .204)$  to give

$$a_{1n} = \frac{9 + KK_t}{(10 + KK_a)^2} = .575$$

$$a_{0n} = \frac{K}{(10 + KK_a)^2} = .204$$

Since  $\omega_{bn} = .7$  is chosen the normalizing factor  $a_3 = 10 + KK_a$  is determined from the relation

$$\omega_b = (10 + KK_a) \omega_{bn} = 10$$

from which  $10 + KK_a = 28.6$  is obtained. Substituting this value of the normalizing factor into the denominators of the two equations for the coordinate of the selected M point above,  $K, K_t$  and then  $K_a$  are determined and the solutions are

$$K = 4675$$

$$K_t = 0.0987$$

$$K_a = 0.00388$$



## 2-6. Frequency response of the fourth order systems.

The  $\Gamma_0$  curve of a fourth order system can be normalized so that a single curve can represent the  $\Gamma_0$  curve of any fourth order system.

Suppose a fourth order characteristic equation is:

$$F(S) = S^4 + a_3 S^3 + a_2 S^2 + a_1 S + a_0 = 0$$

then the  $\Gamma_0$  curve is defined by

$$\begin{aligned} B_1 &= a_3 \omega^2 \\ B_0 &= a_2 \omega^2 - \omega^4 \end{aligned} \quad (2-34a)$$

Equation (2-34a) represents a concave down parabola whose axis of symmetry is  $B_1 = a_2 a_3 / 2$  with vertex at  $(a_2 a_3 / 2, a_2^2 a_3^2 / 4)$ .  $B_0 = 0$  occurs at  $\omega = 0$  and  $\omega = a_2^{1/2}$ .

Assume a normalized frequency  $\omega_{tn}$  such that  $\omega = a_2^{1/2} \omega_{tn}$  then

Equation (2-34a) becomes

$$\begin{aligned} B_1 &= a_2 a_3 \omega_{tn}^2 \\ B_0 &= a_2^2 (\omega_{tn}^2 - \omega_{tn}^4) \end{aligned} \quad (2-35)$$

Let  $B_{1n} = B_1 / a_2 a_3$  and  $B_{0n} = B_0 / a_2^2$  then Equation (2-35) can be written as

$$\begin{aligned} B_{1n} &= \omega_{tn}^2 \\ B_{0n} &= \omega_{tn}^2 - \omega_{tn}^4 \end{aligned} \quad (2-36)$$

Equation (2-36) is exactly the same equation for the  $\Gamma_0$  curve as is associated with a characteristic equation whose coefficients of the  $S^4$ ,  $S^3$  and  $S^2$  terms are unity.

If a curve defined by Equation (2-36) is constructed, then with proper scaling factors of coordinates and frequencies, it will represent exactly the same curve represented by Equation (2-34a) and thus  $\Gamma_0$  curves of fourth order systems are normalized.



The normalizing property of the  $\Gamma_0$  curve for fourth order systems can be used to obtain the frequency response of any fourth order system. Suppose a fourth order system closed loop transfer function

$$\frac{C}{R}(S) = \frac{b_3 S^2 + b_1 S + a_0}{S^4 + a_3 S^3 + a_2 S^2 + a_1 S + a_0} \quad (2-37)$$

which has a second order polynomial in the numerator. For the case of a fourth order system with third order numerator, add  $b_3 S$  to the numerator of Equation (2-37). By using a transformation or frequency scaling

$$S = a_2^{1/2} s$$

Equation (2-37) can be written as

$$\frac{C}{R}(s) = \frac{b_3 a_2 s^2 + b_1 a_2^{1/2} s + a_0}{a_2^2 s^4 + a_3 a_2^{3/2} s^3 + a_2^2 s^2 + a_1 a_2^{1/2} s + a_0}$$

or

$$\frac{C}{R}(s) = \frac{b_3 a_2^{-1} s^2 + b_1 a_2^{-3/2} s + a_0 a_2^{-2}}{s^4 + a_3 a_2^{-1/2} s^3 + s^2 + a_1 a_2^{-3/2} s + a_0 a_2^{-2}} \quad (2-38)$$

The systems represented by Equation (2-37) and (2-38) are equivalent since the transformation  $S = a_2^{1/2} s$  is one to one. Note in Equation (2-38) that the coefficients of the  $s^2$  and  $s^4$  terms in the denominator are made unity, and also, the coefficients in the numerator and denominator of Equation (2-38), by comparing those of same power terms, are scaled by the same amount. For instance,  $b_1$  and  $a_1$  both are multiplied by  $a_2^{-3/2}$ , and likewise for  $a_2$  and  $b_2$ , etc.

Let	$a_{0n} = a_0 / a_2^2$	
	$a_{1n} = a_1 / a_2 a_3$	
	$a_{2n} = a_2 / a_2 = 1$	
	$a_{3n} = a_3 / a_3 = 1$	(2-39)
	$b_{1n} = b_1 / a_3 a_2$	
	$b_{2n} = b_2 / a_2$	
	$a_{2t} = a_2^{1/2} / a_3$	



then Equation (2-38) is

$$\frac{C}{R}(s) = \frac{b_{2n}s^2 + a_{2t}^{-1/2} b_{1n}s + a_{0n}}{s^4 + a_{2t}^{-1/2} s^3 + s^2 + a_{2t}^{-1/2} a_{1n}s + a_{0n}} \quad (2-38a)$$

By substituting  $s = j\omega_{tn}$ , and noting that  $B_{1n} = \omega_{tn}^2$ ,  $B_{0n} = \omega_{tn}^2 - \omega_{tn}^4$  which are the equations defining the normalized  $\Gamma_o$  curve for fourth order systems, the magnitude of the frequency response is given by

$$\left| \frac{C}{R}(j\omega_{tn}) \right| = \frac{|(a_{0n} - b_{2n}\omega_{tn}^2) + j a_{2t} b_{1n} \omega_{tn}|}{|(a_{0n} - B_{0n}) + j a_{2t}^{-1/2} \omega_{tn} (a_{1n} - B_{1n})|} \quad (2-40)$$

thus the frequency response of fourth order systems is represented in terms of normalized quantities.

The methods described in Sections 2-1, 2-2, 2-3 and 2-4 can be applied to get the frequency response of the fourth order system. The frequency response of the original system represented by Equation (2-37) can be obtained from

$$\left| \frac{C}{R}(j\omega) \right| = \left| \frac{(a_0 - b_2\omega^2) + j b_1 \omega}{(a_0 - B_0) + j \omega (a_1 - B_1)} \right| \quad (2-41)$$

which results from applying Equation (2-6) to the system (2-37). The advantage of using normalized Equation (2-40) over the original Equation (2-41) is that, since the normalized  $\Gamma_o$  curve for any fourth order system is a fixed curve, it is not required to plot the  $\Gamma_o$  curve for each system.

Knowing how to interpret Equation (2-41) on the  $B_0$  vs  $B_1$  plane which is associated with the original system, Equation (2-40) can be interpreted on the normalized  $B_{0n}$  vs  $B_{1n}$  plane without difficulty.



The quantities or coordinates in the normalized and unnormalized systems or Mitrovic's plane are related by Equation (2-39) plus the frequency relation

$$\omega = \bar{a}_2^{1/2} \omega_{tn}$$

The M point is located on the normalized  $B_{0n}$  vs  $B_{1n}$  plane by the normalized coordinate  $(a_{1n}, a_{0n})$ .

The imaginary term in the denominator of Equation (2-40) can be represented in the same way as for Equation (2-41), namely by  $\omega(a_1 - B_1)$  plot; however, it must be multiplied by  $a_{2t}^{-1/2}$  to get the  $a_{1t}^{-1/2} \omega_{tn}(a_{1n} - B_{1n})$  plot, and likewise for the imaginary term in the numerator of Equation (2-41) if a system of interest has  $b_1$  other than zero.

In Fig. 2-23, the normalized  $\Gamma_o$  curve for fourth order systems is plotted on the  $B_{0n}$  vs  $B_{1n}$  plane with the same scales for both axes. Any point  $M_n(a_{1n}, a_{0n})$  located on the  $B_{0n}$  vs  $B_{1n}$  plane is equivalent to the point  $M(a_1, a_0)$  on the  $B_0$  vs  $B_1$  plane where  $a_{1n}$ ,  $a_1$  and  $a_{0n}$ ,  $a_0$  are related by Equation (2-39). Any frequency  $\omega_{tn}$  on the normalized  $\Gamma_o$  curve is equal to  $\omega$  divided by  $a_2^{1/2}$  where  $\omega$  is the frequency on the unnormalized  $\Gamma_o$  curve at the point which is equivalent to the  $\omega_{tn}$  point on the normalized  $\Gamma_o$  curve.

The following table is a comparison of the  $\Gamma_o$  curves for normalized and unnormalized systems:

	Normalized $\Gamma_o$ curve	Unnormalized $\Gamma_o$ curve
Axis of symmetry	$B_{1n} = 0.5$	$B_1 = .5 a_2 a_3$
Coordinates of vertex	$(.5, .25)$	$(.5 a_2 a_3, .25 a_2^2)$
Frequency at vertex	$\omega_{tn} = .707$	$\omega = .707 \sqrt{a_2}$
Freq. at $B_1$ intersect	$\omega_{tn} = 0, \omega_{tn} = 1$	$\omega = 0, \omega = \sqrt{a_2}$
Coordinates of $B_1$ intersect	$(0, 0), (1, 0)$	$(0, 0), (0, a_2 a_3)$



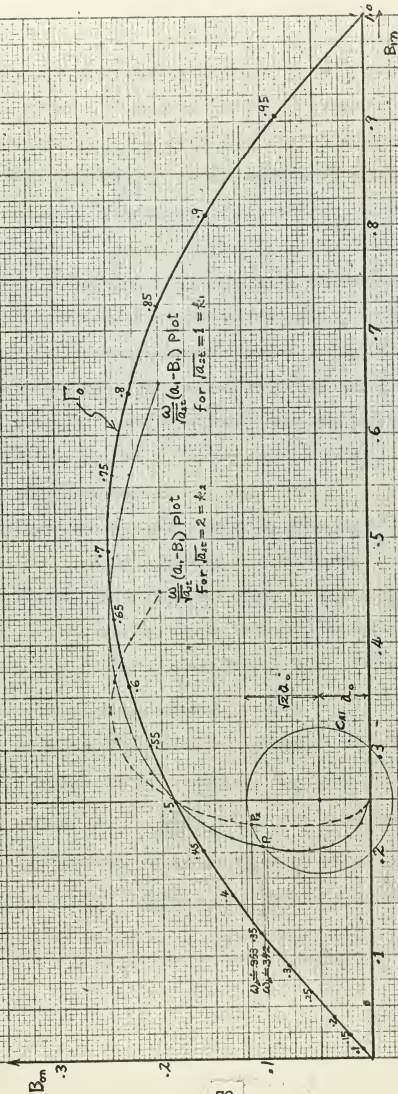


Fig. 2-23 Normalized  $\Gamma_0$  curve for 4th order systems and illustrating the effect of  $a_2$  on  $\omega_{bn}$



The equation defining constant bandwidth curves of fourth order systems on the normalized  $B_0$  vs  $B_1$  plane contains a parameter  $a_{2t}$ , therefore it is not a practical way to try to provide a universal chart of constant bandwidth curves of fourth order systems, because for each  $a_{2t}$  there results a family of constant bandwidth curves. However, it would be worthwhile to investigate the effect of varying  $a_{2t}$  on bandwidth, since if the effect of  $a_{2t}$  on bandwidth is known, then if a constant bandwidth curve is drawn or bandwidth is determined for some  $a_{2t}$  then it may be possible to get some information about bandwidth for another  $a_{2t}$ .

Assume two fourth order systems such that when they are normalized they result in identical system equations except they have different  $a_{2t}$ 's, namely  $a_{2t}^{\frac{1}{2}} = k_1$  for one of the two systems and  $a_{2t}^{\frac{1}{2}} = k_2$  for the other where  $k_1$  and  $k_2$  are constants,  $k_1 < k_2$ .

The two systems have a common M point on the normalized  $B_0$  vs  $B_1$  plane. Suppose this M point is located as shown in Fig. 2-23. Since the normalized  $\Gamma_0$  curve and M point are common to both systems, the  $\omega_{tn}(a_{1n} - B_{1n})$  plot will also be common, but each system will have a different  $\frac{\omega_{tn}}{\sqrt{a_{2t}}}(a_{1n} - B_{1n})$  plot as shown in Fig. 2-23. ( $k_1 = 1$  and  $k_2 = 2$  are assumed for a simple illustration.)

Since the bandwidth of a system is obtained at the point where some magnitude circle intersects the  $\frac{\omega_{tn}}{\sqrt{a_{2t}}}(a_{1n} - B_{1n})$  plot, and noting that the smaller  $a_{2t}$  causes the  $\frac{\omega_{tn}}{\sqrt{a_{2t}}}(a_{1n} - B_{1n})$  plot to be displaced farther from the  $B_{1n} = a_{1n}$  line than the larger  $a_{2t}$  does, any circle centered at the M point intersects the  $\frac{\omega_{tn}}{\sqrt{a_{2t}}}(a_{1n} - B_{1n})$  plot at lower frequency for the smaller  $a_{1t}$ . For example in Fig. 2-23, if the system has no zeros



in its closed loop transfer function, then the magnitude circle  $C_{Ab}$  to determine bandwidth has radius equal to  $\sqrt{2} a_{0n} = .0707$  and it intersects with the two  $\frac{\omega_{bn}}{\sqrt{a_{2t}}}(a_{1n} - B_{1n})$  plots at  $P_1$  and  $P_2$  to give the bandwidths

$$\omega_{bn} = .342 \quad \text{for} \quad \sqrt{a_{2t}} = 1$$

$$\omega_{bn} = .35 \quad \text{for} \quad \sqrt{a_{2t}} = 2 \quad \text{as shown.}$$

It is obvious the above reasoning holds for systems whose closed loop transfer function has any number of zeros, thus a conclusion is obtained as follows:

The effect of  $a_{2t}$  on bandwidth of a normalized fourth order system is to increase the normalized bandwidth with increasing  $a_{2t}$ .

It must be remembered that the normalized bandwidth  $\omega_{bn}$  is related with unnormalized bandwidth  $\omega_b$  by

$$\omega_b = \sqrt{a_{2t}} \omega_{bn}$$

As a simple application of the above conclusion, refer to Fig. 2-13. In Fig. 2-13, the hyperbolas, the constant bandwidth curves are for a system

$$\frac{C}{R}(S) = \frac{a_0}{S^4 + S^3 + S^2 + a_1 S + a_0}$$

This system can be regarded as some system which normalizes so that the coefficient of the  $S^3$  term is unity with  $a_{2t} = 1$ . In other words, if for a normalized system

$$\frac{C}{R}(S) = \frac{a_{0n}}{S^4 + \frac{1}{\sqrt{a_{2t}}} S^3 + S^2 + \frac{a_{1n}}{\sqrt{a_{2t}}} S + a_{0n}}$$

if constant  $\omega_{bn}$  curves are plotted with  $a_{2t} = 1$ , then there will result exactly the same configuration of curves as in Fig. 2-13. For the



$M(a_{1n}, a_{0n})$  point as indicated on the curve  $C_{.5}$ ,  $\omega_{bn}$  is known to be .5. If there is another normalized system with the same M point but with  $a_{2t}$  slightly different from 1, then  $\omega_{bn}$  of this system will not be far off from .5 in the well-known direction of deviation from .5. Same reasoning can be made if the M point is located anywhere on the  $B_{0n}$  vs  $B_{1n}$  plane.

The following numerical examples are to illustrate using the normalized  $\Gamma_0$  curve in obtaining frequency response of fourth order systems.



### Example 2.6

Given a closed loop system transfer function

$$\frac{C}{R}(S) = \frac{1600}{S^4 + 22S^3 + 172S^2 + 680S + 1600} \quad (2-42)$$

obtain the frequency response (magnitude).

The problem can be worked out without normalizing the systems; however, it will be shown on the normalized  $B_{0n}$  vs  $B_{1n}$  plane.

For the given system, the coefficients are

$$a_0 = 1600$$

$$a_1 = 680$$

$$a_2 = 172$$

$$a_3 = 22$$

By using relations between normalized and unnormalized quantities in Equation (2-39), the coefficients are normalized to

$$a_{0n} = a_0 / a_2^2 = .0542$$

$$a_{1n} = a_1 / a_2 a_3 = .18$$

and  $\sqrt{a_{2n}} = \sqrt{a_2} / a_3 = .595$

The frequencies of the normalized and unnormalized system are related by

$$\omega = \sqrt{a_{2n}} \omega_{tn} = 13.1 \omega_{tn}$$

From Equation (2-40), the frequency response (magnitude) of the given system is, in terms of normalized quantities,

$$\left| \frac{C}{R}(j\omega_{tn}) \right| = \left| \frac{.0542}{(.0542 - B_{0n}) + j \frac{\omega_{tn}}{.595} (.18 - B_{1n})} \right|$$

In Fig. 2-24 the point  $M(a_{1n}, a_{0n}) = M(.18, .0542)$  is located and the lines  $B_{1n} = a_{1n} = .18$ ,  $B_{0n} = a_{0n} = .0542$ ,  $B_{0n} = (1 + \sqrt{2}) a_{0n} = .131$



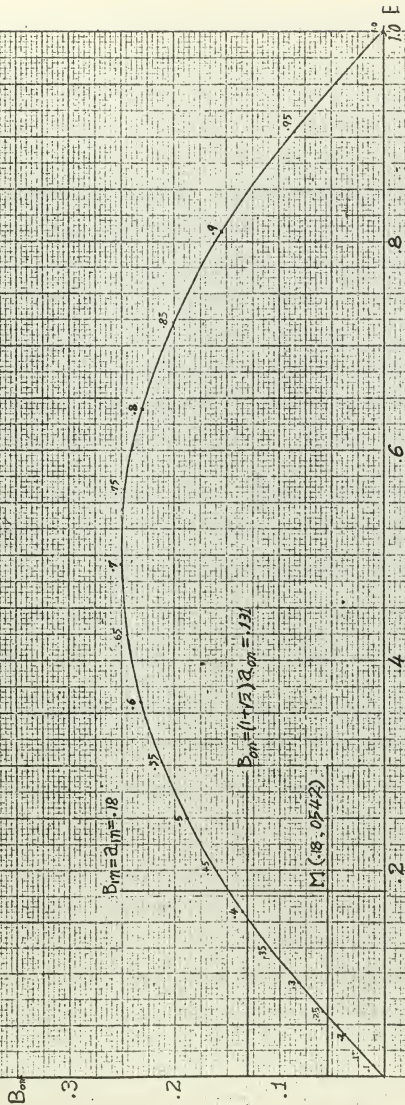


Fig. 2-24 The normalized  $\bar{\sigma}$  curve and M point configuration for the system in Example 2.6.



are drawn on the normalized  $B_0$  vs  $B_1$  plane.

Since the line  $B_{0n} = (1 + \sqrt{2})a_{0n} = .131$  crosses the  $\Gamma_0$  curve at the point where the frequency is slightly less than  $\omega_{tn} = .4$ , and since the system has no zeros, the bandwidth  $\omega_{bn}$  must be less than .4. From the  $\Gamma_0$  curve and M point configuration on the normalized  $B_0$  vs  $B_1$  plane as shown in Fig. 2-24, it is seen that the first one-half part of the curve is sufficient to see the frequency response of the system.

In Fig. 2-25, the  $\Gamma_0$  curve, M point and the  $B_{1n} = a_{1n}$ ,  $B_{0n} = a_{0n}$  and  $B_{0n} = (1 + \sqrt{2})a_{0n}$  lines are drawn to expanded coordinate scales. Frequency response is represented in the same way as in Fig. 2-8, the only different thing from Fig. 2-8 is the  $\frac{\omega_{tn}}{\sqrt{a_{2t}}}(a_{1n} - B_{1n})$  plot.

As illustrated previously, the  $\frac{\omega_{tn}}{\sqrt{a_{2t}}}(a_{1n} - B_{1n})$  plot is constructed such that the horizontal distance from the line  $B_{1n} = a_{1n} = .18$  to any point on the  $\frac{\omega_{tn}}{\sqrt{a_{2t}}}(a_{1n} - B_{1n})$  plot is equal to the horizontal distance from the line  $B_{1n} = a_{1n} = .18$  to the  $\Gamma_0$  curve multiplied by  $\frac{\omega_{tn}}{\sqrt{a_{2t}}} = \frac{\omega_{tn}}{.595}$ . The  $\frac{\omega_{tn}}{\sqrt{a_{2t}}}(a_{1t} - B_{1t})$  plot bears the same frequency as  $\Gamma_0$  if the ordinates are the same as in the  $\omega(a_1 - B_1)$  plot.

Some magnitude circles are drawn to show the magnitude  $A = \left| \frac{C}{R} \int \omega_{tn} \right|$  at some frequencies. The magnitude circle tangent to the  $\frac{\omega_{tn}}{\sqrt{a_{2t}}}(a_{1n} - B_{1n})$  plot which is for resonant point touches to the  $\frac{\omega_{tn}}{\sqrt{a_{2t}}}(a_{1n} - B_{1n})$  plot at  $\omega_{tn} = .24$  with radius equal to .05 to give

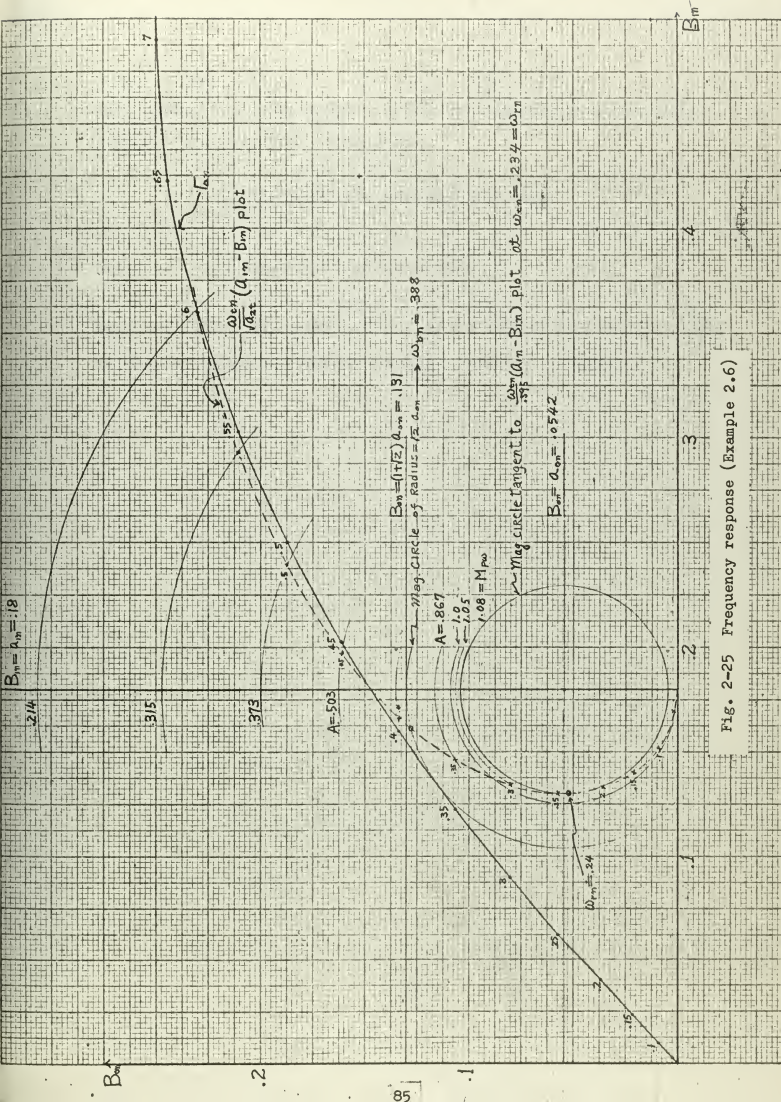
$$\text{the resonance frequency } \omega_{rn} = .24 \text{ or } \omega_r = (13.1) \omega_{rn} = 3.14$$

$$\text{the resonance peak } M_{p\omega} = .0542 / .05 = 1.08$$

The magnitude circle with radius equal to  $\sqrt{2} a_0 = .765$  crosses the

$$\frac{\omega_{tn}}{\sqrt{a_{2t}}}(a_{1n} - B_{1n}) \text{ plot at } \omega_{tn} = .388 \text{ to give the bandwidth}$$







$$\omega_{bn} = .388 \quad \text{or} \quad \omega_b = (13.1) (.388) = 5.09$$

Substituting  $S = j\omega$  in Equation (2-42), the magnitude for  $\omega = \omega_r = 3.14$  and  $\omega = \omega_b = 5.09$  calculate to

$$\left| \frac{C}{R}(j3.14) \right| = 1.1 \text{ for resonance peak } M_{pw}$$

and  $\left| \frac{C}{R}(j5.09) \right| = .71 \text{ for bandwidth}$

Magnitude circles are drawn in Fig. 2-25 for several frequencies and magnitudes are indicated, thus the frequency response is obtained.



### Example 2.7

Given an open loop transfer function for a unity feedback control system as

$$G(s) = \frac{31.6(s+1)(s+5)}{s(s+.333)(s+2)(s+10)}$$

determine the bandwidth of the closed loop system.

The closed loop system transfer function is

$$\frac{C}{R}(s) = \frac{31.6s^2 + 190s + 158}{s^4 + 12.333s^3 + 55.6s^2 + 196.66s + 158} \quad (2-43)$$

which has a second order numerator.

The quantities to be normalized to work on the normalized  $B_0$  vs  $B_1$  plane are, using the relations in Equation (2-39),

$$a_{0n} = 158/55.6 = .0512$$

$$a_{1n} = 196.66/(12.333)(55.6) = .286$$

$$b_{1n} = 190/(12.333)(55.6) = .277$$

$$b_{2n} = 31.6/55.6 = .568$$

$$\sqrt{a_{2t}} = \sqrt{55.6/12.333^2} = .603$$

$$\sqrt{a_2} = 7.44$$

$$\omega = 7.44 \omega_{tn}$$

The frequency response of the system in terms of normalized quantities is defined by, using Equation (2-40),

$$\left| \frac{C}{R}(j\omega_{tn}) \right| = \left| \frac{(.0512 - .568\omega_{tn}^2) + j \frac{\omega_{tn}}{.603}(.277)}{(.0512 - B_{0n}) + j \frac{\omega_{tn}}{.603}(.286 - B_{1n})} \right|$$

For the bandwidth

$$\left| (.0512 - B_{0n}) + j \frac{\omega_{tn}}{.603}(.286 - B_{1n}) \right| = \sqrt{2} \left| (.0512 - .568\omega_{tn}^2) + j \frac{\omega_{tn}}{.603}(.277) \right| \quad (2-44)$$



The left side of Equation (2-44) can be represented by the usual

$\frac{\omega_{tn}}{.602} (.286 - B_{1n})$  plot. On the right side of Equation (2-44), both real and imaginary terms are frequency dependent. As illustrated in Section 2-3, the real term with the factor  $\sqrt{2}$  is represented on the  $B_{1n} = a_{1n} = .286$  line and the imaginary term with the factor  $\sqrt{2}$  is represented on the  $B_{0n} = a_{0n} = .0512$  line. Those are shown in Fig. 2-26.

In Fig. 2-26, the normalized  $\Gamma_0$  curve with the point M(.286, .0512) is shown. On  $B_{1n} = .286$  line,  $\sqrt{2} b_{2n} \omega_{tn}^2 = .805 \omega_{tn}^2$  is calibrated as follows. The point corresponding to  $\omega_{tn} = 0$  which is located below  $B_{1n}$  axis is at the distance of  $\sqrt{2} a_{0n} = .0723$  from the M point. From this point up, the  $B_{1n} = .286$  line is labeled with  $\omega_{tn}$  such that the distance from the  $\omega_{tn} = 0$  point to a  $\omega_{tn} = \omega_{t1}$  point is equal to

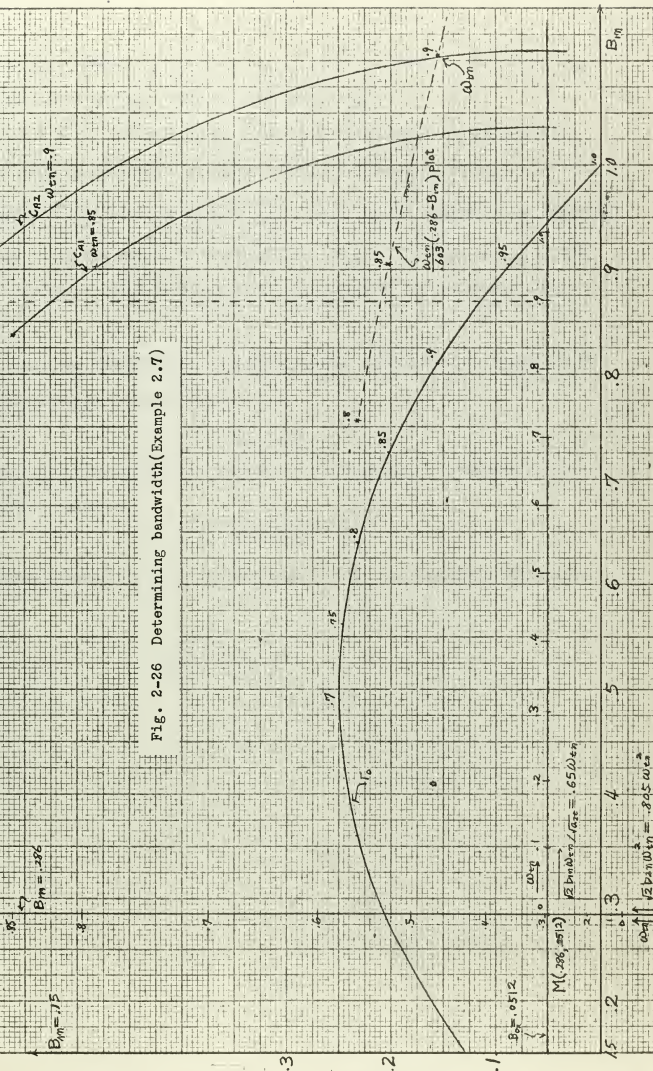
$\sqrt{2} b_{1n} \omega_{t1}^2 = .805 \omega_{t1}^2$  so that at  $\omega_{tn} = \omega_{t1}$ , the distance from the M point to the  $\omega_{tn} = \omega_{t1}$  point on the  $B_{1n} = .286$  line is equal to

$$|\sqrt{2} (a_{0n} - b_{1n} \omega_{t1}^2)| = |.0723 - .805 \omega_{t1}^2|$$

For example at  $\omega_{tn} = .9$ ,  $\sqrt{2} (a_{0n} - b_{1n} \omega_{tn}^2)$  is equal to the distance from the M point to the  $\omega_{tn} = .9$  point on the  $B_{1n} = .286$  line, which is equal to .73.

The imaginary term in the right side of Equation (2-44) is represented on the  $B_{0n} = a_{0n} = .0512$  line as follows. On the  $B_{0n} = a_{0n} = .0512$  line, each point to the right of the M point bears frequency  $\omega_{tn}$  such that a point corresponding to  $\omega_{tn} = \omega_{t1}$  on the  $B_{0n} = .0512$  line is located at a distance of  $\sqrt{2} b_{1n} \omega_{t1} / \sqrt{a_{2t}} = .65 \omega_{t1}$ . For example, at  $\omega_{tn} = .9$  on the  $B_{0n} = .0512$  line, the distance from the M point is







$(.65)(.9) = .585$ , thus the line  $B_{0n} = .0512$  has linearly displaced  $\omega_{tn}$  whereas the line  $B_{1n} = .286$  has quadratic  $\omega_{tn}$ .

With those two lines, the quantity on the right side of Equation (2-44) can be read off at any frequency. For example, at  $\omega_{tn} = .9$  the quantity  $\sqrt{2} \left| (.512 - .568 \omega_{tn}^2) + j \frac{\omega_{tn}}{.603} (.277) \right|$  is the length of the diagonal of the rectangle formed by the point M,  $\omega_{tn} = .9$  points on  $B_{1n} = .286$  line and  $B_{0n} = .0512$  line, which is the radius of the magnitude circle  $C_{A2}$ .

Then the bandwidth is determined in the same way as in Example 2.3. In this example, as seen in Fig. 2-26,

$$\omega_{bn} = .9$$

Noting that frequencies are scaled by  $\omega = 7.44 \omega_{tn}$ , bandwidth of the original system is

$$\omega_b = 6.7$$

By substituting  $S = j 6.7$  in Equation (2-43), the magnitude calculates to

$$\left| \frac{C}{R}(j 6.7) \right| = .707$$



## 2-7. Evaluation of phase angle.

In the previous sections the magnitudes of the frequency response of a system are evaluated on the  $B_0$  vs  $B_1$  plane. Since the numerator and the denominator of the equation defining frequency response can be represented as vectors on the  $B_0$  vs  $B_1$  plane, it is obvious that the phase angles in the frequency response of a system can also be evaluated on the  $B_0$  vs  $B_1$  plane.

In this section the method for evaluating the phase angles in the frequency response of a system is illustrated with some of the numerical examples which are considered in the previous sections of the magnitude evaluation.

Again writing the general equation for the frequency response of a system as

$$\frac{C}{R}(j\omega) = \frac{N(j\omega)}{(a_0 - B_0) + j\omega(a_1 - B_1)}$$

the phase of the frequency response is

$$\angle \frac{C}{R}(j\omega) = \angle N(j\omega) - \angle ((a_0 - B_0) + j\omega(a_1 - B_1)) \quad (2-45)$$

Since the vectors  $N(j\omega)$  and  $(a_0 - B_0) + j\omega(a_1 - B_1)$  are measured from the M point and the real and imaginary parts of the vectors are represented on the  $B_0$  vs  $B_1$  plane in such orientations that they are parallel to the  $B_0$  and  $B_1$  axes respectively, a proper reference in measuring the angles must be set up.

Consider the angle  $\angle((a_0 - B_0) + j\omega(a_1 - B_1))$  with the aid of Fig. 2-27. The  $\omega(a_1 - B_1)$  plot is constructed as illustrated in the previous sections with the  $\Gamma_0$  curve and the M point configuration as shown. Select a point such as P in Fig. 2-27 at frequency of  $\omega_1$  and



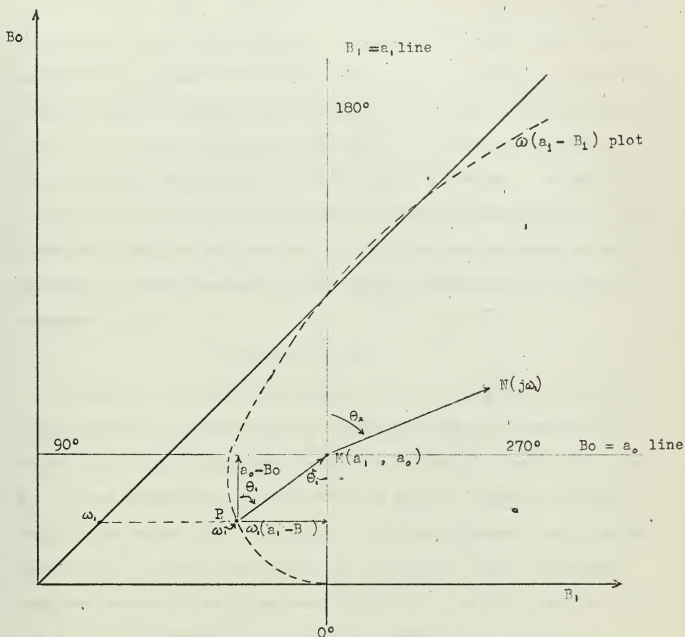


Fig. 2-27. Evaluation of the phase angles in the frequency response.



draw the vector  $\vec{PM}$ . The components of this vector define the real and imaginary terms  $(a_0 - B_0)$  and  $\omega_1(a_1 - B_1)$  in magnitude and sign. The horizontal component  $\omega_1(a_1 - B_1)$  is the imaginary part and positive when directed to the right. The vertical component  $(a_0 - B_0)$  is the real part and positive when directed upward. Considering the signs of the component vectors, the angle of the vector  $\vec{PM} = (a_0 - B_0) + j\omega_1(a_1 - B_1)$  is the angle  $\theta_1$  which is measured from the vector  $(a_0 - B_0)$  to  $\vec{PM}$ , and this is equal to the angle measured from the line  $B_1 = a_1$  below the M point to  $\vec{MP}$  and positive when measured to the clockwise direction. The angle of the vector  $N(j\omega)$  is naturally  $\theta_2$  which is measured from the line  $B_1 = a_1$  above the M point to the vector  $N(j\omega)$  with the positive direction as indicated, thus the phase angle of the system in the frequency response at frequency  $\omega_1$  is

$$\angle \frac{C}{R}(j\omega_1) = \theta_2 - \theta_1$$

It must be noted that the reference line of the angle of  $N(j\omega)$  depends upon how the vector  $N(j\omega)$  is represented and it is seen that the vector  $N(j\omega)$  can be represented either to the right or left of the line  $B_1 = a_1$  and either above or below the line  $B_0 = a_0$ ; however, for the angle of the vector  $(a_0 - B_0) + j\omega(a_1 - B_1)$ , the reference line is always the line  $B_1 = a_1$  below the M point and the positive angle is measured from the reference line to the vector  $\vec{MP}$  in the clockwise direction, thus the  $0^\circ$ ,  $90^\circ$ ,  $180^\circ$  and  $270^\circ$  positions are as indicated.

Evaluation of phase angles in the frequency response of an open loop system is also possible as the case of obtaining magnitude in the frequency response. The following examples illustrate evaluation of phase angles in the frequency response for closed loop systems as well as for open loop systems.



### Example 2.8

For the system equation in Example 2.1 which is rewritten below

$$G(s) = \frac{4}{s(s+1)(s+2)}$$

obtain the phase characteristics in the frequency responses for both closed loop and open loop systems.

#### Closed loop system

The closed loop system has the transfer function

$$\frac{C}{R}(s) = \frac{4}{s^3 + 3s^2 + 2s + 4}$$

in which the numerator polynomial is of order zero and the angles

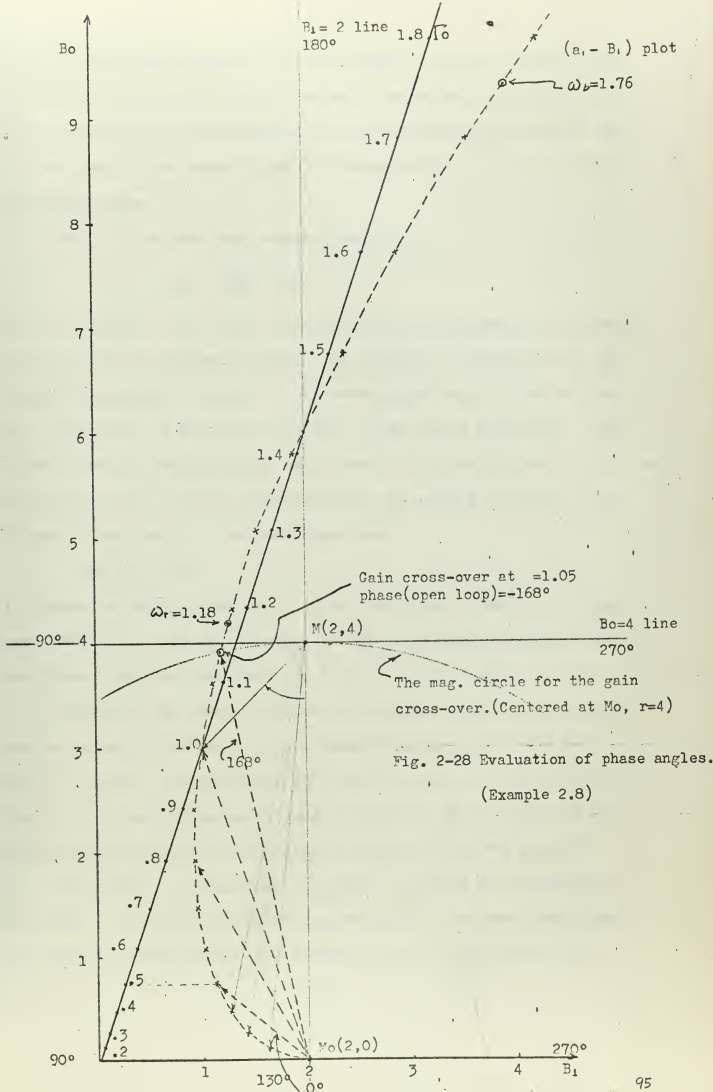
$\angle N(j\omega) = 0$  for all frequencies, therefore only the angles

$\angle((a_0 - b_0) + j\omega(a_1 - b_1))$  are frequency dependent.

In Fig. 2-28, the  $\Gamma_0$  curve and the  $\omega(a_1 - b_1)$  plot are duplicated from Fig. 2-8 which is for obtaining the magnitude of the closed loop frequency response of the same system.

The reference angles  $0^\circ$ ,  $90^\circ$ ,  $180^\circ$  and  $270^\circ$  are indicated. The radial vectors directed from the M point to the  $\omega(a_1 - b_1)$  plot give the angles of the denominator of the closed loop frequency response equation which is the vector  $(4 - b_0) + j\omega(2 - b_1)$ . Noting that the angles measured from the  $0^\circ$  reference line to the vectors are of the denominator and also noting that the angles of the numerator are zero degrees for all frequencies, the phase of the closed loop system at each frequency is the negative of the angle of the vector so measured, thus at  $\omega = 0$  the phase is zero degrees and as  $\omega$  increases the phase increases in the negative direction. For example, at  $\omega = 1.0$  the angle of the vector measures  $45^\circ$  and the phase of the system is  $-45^\circ$ . The phase assumes  $-180^\circ$  at







$\omega = 1.4$  and at this point the  $\Gamma_o$  curve,  $\omega(a_1 - B_1)$  plot and the line  $B_1 = a_1 = 2$  intersect. If a protractor is overlapped on the figure such that the center point of the protractor coincides with the M point, then one can read off the phase at each  $\omega$  conveniently.

#### Open loop system

Rewriting the open loop transfer function as

$$G(S) = \frac{4}{S^3 + 3S^2 + 2S}$$

one can construct the  $\Gamma_o$  curve associated with the denominator polynomial of the open loop transfer function. By comparing the equation with the closed loop transfer function, it is obvious that the  $\Gamma_o$  curve for the open loop system is identical with that of the closed loop system. The  $M_o$  point defined analogously as the M point is, by noting that the coefficient of the  $S^0$  term in the denominator polynomial of the open loop system is zero, at  $B_1 = 2$  on the  $B_1$  axis and

$$M_o = M_o(2, 0)$$

is located in Fig. 2-28 as shown. Since the  $M_o$  and M points are on the same line  $B_1 = 2$ , and the  $\Gamma_o$  curves for the two transfer functions (open and closed loop) are identical, the  $\omega(a_1 - B_1)$  plot is not altered.

By applying the same principles to the open loop system the phase can be evaluated with the vectors directed from the  $M_o$  point to the  $\omega(a_1 - B_1)$  plot. Naturally the  $90^\circ$ --- $270^\circ$  line must be shifted down to the  $B_1$  axis. Noting that as  $\omega$  tends to zero the  $\omega(a_1 - B_1)$  plot approaches the  $M_o$  point from the direction of the origin the phase at  $\omega = 0$  is  $-90^\circ$ . As  $\omega$  increases the phase increases in the negative direction. Some vectors representing the phase of the open loop system are drawn with dotted lines, for example at  $\omega = .5$  the angle of the



vector measures  $130^\circ$  to give a  $-130^\circ$  phase.

At  $\omega = 1.42$  which is the point where the phase of the closed loop system is  $-180^\circ$ , the open loop system also assumes  $-180^\circ$  phase and it is interesting to note that this occurs for any system which has a frequency independent numerator polynomial.

As  $\omega$  tends to infinity, from the behavior of the  $\Gamma_o$  curve and the  $\omega(a_1 - B_1)$  plot it is seen that the phase approaches  $-270^\circ$  which is the expected phenomenon.

In the Bode diagram manipulation, the gain crossover frequency and the phase at that point are used for the stability check and as a figure of merit in a design problem. The gain crossover point which is the zero db or unity magnitude point can be found simply by drawing a magnitude circle with radius equal to  $a_0 = 4$  centered at the  $M_0$  point, which is to account the magnitude of the open loop system, thus the gain crossover frequency is found to be 1.05 and the phase at this point is  $-168^\circ$  to give  $+12^\circ$  phase margin. This phase margin indicates a stable but too oscillatory system which checks with the  $\Gamma_o$  curve and the M point configuration. The gain margin can be found by drawing another magnitude circle centered at the  $M_0$  point passing through the  $-180^\circ$  phase point.



### Example 2.9

For the system in Example 2.2

$$G(s) = \frac{121000(s+47)}{s(s+1.1)(s+1670)}$$

the transformed closed loop system equation is, by using the transformation  $s = 100s$

$$\frac{C}{R}(s) = \frac{12.1s + 4.48}{s^3 + 16.71s^2 + 12.267s + 4.48}$$

In order to obtain the phase characteristic of the closed loop system the  $\Gamma_0$  curve and the  $\omega_t(a_{1t} - b_{1t})$  plot of Fig. 2-9 are reproduced in Fig. 2-29.

The angles of the denominator vectors  $\theta_1$  are measured in a similar way as in the previous example. Noting that the system has the first order numerator and the vector  $N(j\omega_t)$  at each  $\omega_t$  is represented by a vector directed from the M point to the line  $B_{0t} = 2a_{0t}$  (the two vectors for  $\omega_t = .2$  and  $\omega_t = .3$  are shown in the figure with dotted lines) the angle  $\angle N(j\omega_t)$  at each  $\omega_t$  is measured from the line  $B_{1t} = a_{1t}$  above the M point to the vector  $N(j\omega_t)$  at each  $\omega_t$ . For example at  $\omega_t = .2$ ,  $\angle N(j.2) = 29^\circ$ . Denoting the angles of the denominator and the numerator vectors at the same frequency by  $\theta_1$  and  $\theta_2$  respectively, then the phase of the closed loop system  $\angle \frac{C}{R}(j\omega_t) = \theta_2 - \theta_1$ , for example at  $\omega_t = .2$ ,  $\angle \frac{C}{R}(j.2) = 29^\circ - 31.5^\circ = -2.5^\circ$ .

For the open loop system the transformed system equation is

$$G(s) = \frac{12.1s + 4.48}{s^3 + 16.71s^2 + .167s}$$

The  $\Gamma_0$  curve for the open loop system is not changed from that of the closed loop system but the  $M_0$  point is moved to the point  $(.167, 0)$  which is seen from the coefficients of the  $s^1$  and  $s^0$  terms in the



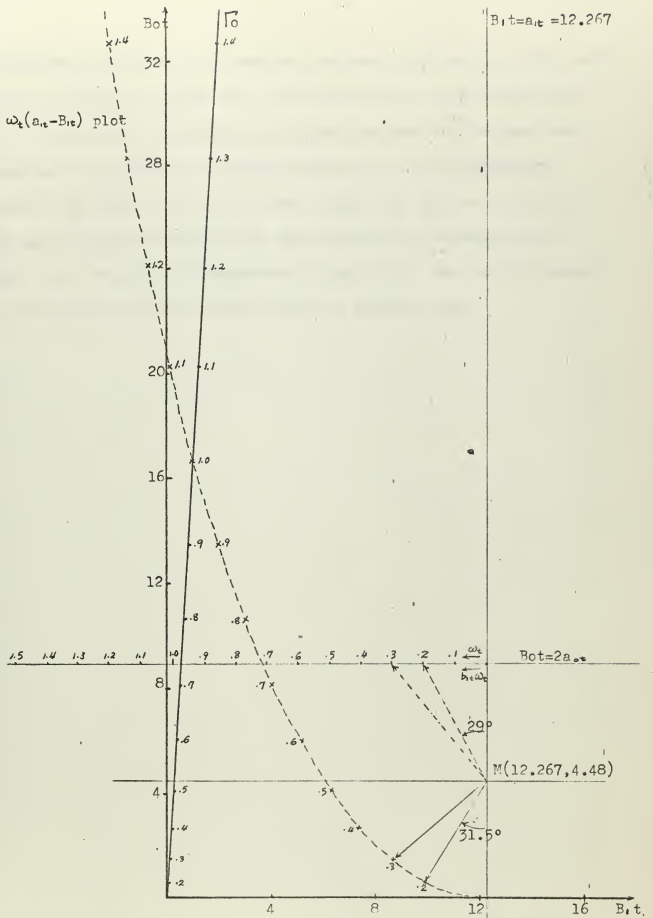


Fig. 2-29 Evaluation of phase angles.

(Example 2.9).



denominator of the open loop transfer function. This implies that a new  $\omega_t(a_{1t} - B_{1t})$  plot for the open loop system must be constructed with  $a_{1t} = .167$  in order to obtain the frequency response of the open loop system and the procedure to construct this plot is straightforward. Since the  $M_0$  point is so close to the origin, the  $\omega_t(.167 - B_{1t})$  plot will not give good resolutions of the angles of the denominator for small  $\omega_t$  if the plot is constructed in Fig. 2-29. This can be overcome by plotting the curves for small  $\omega_t$  on an expanded scale.



### 3. The fourth order system charts.

In references 1 and 2, a universal third order system chart is constructed on the  $B_0$  vs  $B_1$  plane so that a single chart will provide solutions to any third order system problem.

This is done by using a suitable transformation. Any third order system is transformed into a normalized system so that it has unity coefficients of the terms  $s^3$  and  $s^2$  in the characteristic equation to give a set of fixed  $B_0$  vs  $B_1$  curves.

In a fourth order system, assuming unity as coefficient of the  $s^4$  term in its characteristic equation,  $B_0$  vs  $B_1$  curves are changed by changing either  $a_3$  the coefficient of the  $s^3$  term or  $a_2$  the coefficient of the  $s^2$  term.

Since for any polynomial it is possible to make any two of the coefficients unity (or the same numbers), by a suitable linear transformation, in a fourth order system characteristic equation any two of the coefficients of  $s^4$ ,  $s^3$  and  $s^2$  terms which are the coefficients affecting the  $B_0$  vs  $B_1$  curves can be made unity so that all  $B_0$  vs  $B_1$  curves of any fourth order system depend upon only one coefficient.

For each value of the variable coefficient, there results a different set of the Mitrovic's curves. If the sets of the curves for some suitable pertinent values of the variable coefficient are constructed and if the behavior of the set of the curves for the variable coefficient changing from one pertinent value to others is understood, then the sets of the curves may serve as fourth order system charts which can be applied to any fourth order systems.



### 3-1. Transformation of the fourth order system equation.

Suppose a fourth order system characteristic equation is:

$$F(S) = S^4 + a_3 S^3 + a_2 S^2 + a_1 S + a_0 \quad (3-1)$$

and associated  $B_0$  vs  $B_1$  curves are defined by

$$\begin{aligned} B_1 &= a_2 \phi_2(s) \omega_n + a_3 \phi_3(s) \omega_n^2 + \phi_4(s) \omega_n^3 \\ B_0 &= a_2 \omega_n^2 - a_3 \phi_2(s) \omega_n^3 - \phi_3(s) \omega_n^4 \end{aligned} \quad (3-2)$$

Using a transformation or frequency scaling

$$S = a_3 s$$

Equation (3-1) can be written as

$$F(a_3 s) = a_3^4 s^4 + a_3^4 a_3^3 s^3 + a_2 a_3^2 s^2 + a_1 a_3 s + a_0 = 0$$

or by dividing through by  $a_3^4$

$$f(s) = s^4 + s^3 + a_2/a_3^2 s^2 + a_1/a_3^3 s + a_0/a_3^4 = 0$$

$$\text{or} \quad f(s) = s^4 + s^3 + a_{2t} s^2 + a_{1t} s + a_{0t} = 0 \quad (3-3)$$

$$\text{where} \quad a_{2t} = a_2 / a_3^2$$

$$a_{1t} = a_1 / a_3^3$$

$$a_{0t} = a_0 / a_3^4$$

Thus Equation (3-1) is transformed into Equation (3-3) in which the two coefficients of the  $s^4$  and  $s^3$  terms are unity.

Equations (3-1) and (3-3) are equivalent since the transformation  $S = a_3 s$  is linear and one-to-one.

The  $B_0$  vs  $B_1$  curves associated with Equation (3-3) are defined by

$$\begin{aligned} B_{1t} &= a_{2t} \phi_2(s) \omega_{nt} + \phi_3(s) \omega_{nt}^2 + \phi_4(s) \omega_{nt}^3 \\ B_{0t} &= a_{2t} \omega_{nt}^2 - \phi_2(s) \omega_{nt}^3 - \phi_3(s) \omega_{nt}^4 \end{aligned} \quad (3-4)$$



Equations (3-2) and (3-4) are equivalent to each other with the following relations

$$B_{1t} = B_1 / a_3^3$$

$$B_{0t} = B_0 / a_3^4$$

$$\omega_{nt} = \omega_n / a_3$$

Comparing Equations (3-2) and (3-4) which are Mitrovic's equations for a system represented on different coordinate scales, it is seen that studying a fourth order system with the second equation is more convenient than with the first, because in the second, only one variable coefficient  $a_{2t}$  appears whereas in the first the two coefficients  $a_2$  and  $a_3$  appear.

A fourth order system closed loop transfer function represented as

$$\frac{C}{R}(S) = \frac{b_2 S^2 + b_1 S + a_0}{S^4 + a_3 S^3 + a_2 S^2 + a_1 S + a_0} \quad (3-5)$$

transforms into

$$\frac{C}{R}(s) = \frac{b_{2t} s^2 + b_{1t} s + a_{0t}}{s^4 + s^3 + a_{2t} s^2 + a_{1t} s + a_{0t}} \quad (3-6)$$

to which the Mitrovic's curves defined by Equation (3-4) associate.

The coefficients and frequencies of the transformed system are related with those of the original system by

$$b_{2t} = b_2 / a_3^2$$

$$b_{1t} = b_1 / a_3^3$$

$$\omega_{nt} = \omega_n / a_3$$

plus the relations defined as in Equation (3-3) for the coefficients of the denominators.

The velocity constants,  $K_v$  and  $K_{vt}$  of the original and transformed systems respectively are related by



$$K_{vt} = K_v / a_3$$

since 
$$K_v = \frac{a_0}{a_1 - b_1}$$

$$K_{vt} = \frac{a_{0t}}{a_{1t} - b_{1t}} = \frac{a_0 / a_3}{a_1 / a_3 - b_1 / a_3} = \left( \frac{a_0}{a_1 - b_1} \right) / a_3 = K_v / a_3$$

Thus a fourth order system is transformed into a system from which resulting  $B_0$  vs  $B_1$  curves are dependent upon only one coefficient  $a_{2t}$ . Comparing Equations (3-6) and (2-38a), it is noticed that the characteristic equations are of different forms which indicates that the normalized  $\Gamma_0$  curve in Section 2-6 is different from the  $\Gamma_0$  curve resulting from Equation (3-4) and any  $\Gamma_3$  curve defined by Equation (3-4) cannot be plotted on the normalized  $B_0$  vs  $B_1$  plane unless the system Equation (3-6) is transformed into the form of Equation (2-38a) and Equation (3-4) is forced into a form to give the normalized  $\Gamma_0$  curve. If one wishes to work on the normalized  $B_0$  vs  $B_1$  plane with  $\Gamma_3$  curves, then the system Equation (3-5) or (3-6) is transformed into the form of Equation (2-38a) to give the equation defining Mitrovic's curves as

$$\begin{aligned} B_{1n} &= \sqrt{a_{2t}} \phi_2(s) \omega_{nt} + \phi_3(s) \omega_{nt}^2 + \sqrt{a_{2t}} \phi_4(s) \omega_{nt}^3 \\ B_{0n} &= \omega_{nt}^2 - \frac{1}{\sqrt{a_{2t}}} \phi_2(s) \omega_{nt}^3 - \phi_3(s) \omega_{nt}^4 \end{aligned} \quad (3-7)$$

Equation (3-7) gives the normalized  $\Gamma_0$  curve, and the  $\Gamma_3$  curves can be plotted on the normalized  $B_0$  vs  $B_1$  plane; however,  $\sqrt{a_{2t}}$  appears in too many terms and more complicated behavior of the curves is expected. Thus the normalized  $B_0$  vs  $B_1$  plane is not suitable to study fourth order systems. However the normalized curve can do an excellent job in determining absolute stability of a fourth order system.

The coefficient  $a_{2t}$  appearing in Equation (3-4) indicates that  $B_0$  vs  $B_1$  curves for a fourth order system depend upon the value of  $a_{2t}$ ,



therefore it seems worth while to investigate the possible range of  $a_{2t}$ .

The coefficients  $a_2$  and  $a_3$  of Equation (3-1), the characteristic equation of the original system, can be written as

$$a_3 = r_1 + r_2 + r_3 + r_4 \quad (3-8)$$

$$a_2 = r_1 r_2 + r_1 r_3 + r_1 r_4 + r_2 r_3 + r_2 r_4 + r_3 r_4$$

where  $-r_1$ ,  $-r_2$ ,  $-r_3$  and  $-r_4$  are the roots of Equation (3-1).

The roots  $r_1$ ,  $r_2$ ,  $r_3$  and  $r_4$  fit into one of the following three cases:

- 1) All 4 roots are real.
- 2) Two real and one pair of complex conjugate roots.
- 3) Two pairs of complex conjugate roots.

Noting that  $a_{2t} = a_2/a_3^2$ , by using  $a_2$  and  $a_3$  expressed in terms of the roots, it is possible to determine the range of  $a_{2t}$  to cover most of the fourth order systems for Case 1 and Case 2.

#### Case 1

$$\begin{aligned} a_{2t} &= \frac{r_1 r_2 + r_1 r_3 + r_1 r_4 + r_2 r_3 + r_2 r_4 + r_3 r_4}{(r_1 + r_2 + r_3 + r_4)^2} \\ &= \frac{r_1 r_2 + r_1 r_3 + r_1 r_4 + r_2 r_3 + r_2 r_4 + r_3 r_4}{r_1^2 + r_2^2 + r_3^2 + r_4^2 + 2(r_1 r_2 + r_1 r_3 + r_1 r_4 + r_2 r_3 + r_2 r_4 + r_3 r_4)} \end{aligned}$$

$$\text{noting that} \quad r_1^2 + r_2^2 + r_3^2 + r_4^2 > 0$$

$$0 < a_{2t} < .5$$

It can also be seen that for a fourth order system, if its open loop transfer function has a numerator polynomial of order less than two, and if it has all real poles, then

$$a_{2t} < .5$$

In such a system, the coefficients  $a_2$  and  $a_3$  of its characteristic equation can be expressed in terms of the poles of the open loop transfer



function. Those poles of the open loop transfer function replace the roots of the characteristic equation in Equation (3-8) to get the same result.

### Case 2

Let the two real roots be  $r_1$  and  $r_2$ , and the complex conjugate roots  $r_3$  and  $r_4$ .

$$r_3 = \zeta \omega_n + j \sqrt{1-\zeta^2} \omega_n$$

$$r_4 = \zeta \omega_n - j \sqrt{1-\zeta^2} \omega_n$$

and 
$$a_3^2 = (r_1 + r_2 + 2 \zeta \omega_n)^2 = (r_1 + r_2)^2 + 4 \zeta \omega_n (r_1 + r_2) + 4 \zeta^2 \omega_n^2$$

$$a_2 = r_1 r_2 + 2 \zeta \omega_n (r_1 + r_2) + \omega_n^2$$

to compare  $a_3^2$  and  $a_2$

$$a_3^2 - a_2 = r_1^2 + r_1 r_2 + r_2^2 + 2 \zeta \omega_n (r_1 + r_2) + (4 \zeta^2 - 1) \omega_n^2 \quad (3-9)$$

Assuming a stable system, Equation (3-9) is examined for several situations.

If  $r_1, r_2 > \zeta \omega_n$  then

$$a_3^2 - a_2 > (1 + \zeta^2 - 1) \omega_n^2 \quad \text{implies that}$$

if  $\zeta > .302$  then  $a_3^2 - a_2 > 0$  and  $a_{2t} < 1.0$

It must be noted that the inequality above may have considerable tolerance which implies that even for  $\zeta$  some value less than .302  $a_{2t}$  may be less than 1.

Let  $r_2 > r_1$  and by neglecting  $r_1$  in Equation (3-9)

$$a_3^2 - a_2 > r_2^2 + 2 \zeta \omega_n r_2 + (4 \zeta^2 - 1) \omega_n^2 \quad (3-10)$$

For a well behaving system,  $r_2$  is usually greater than  $\zeta \omega_n$  the real part of the complex conjugate roots.



Let  $r_2 = k \xi \omega_n$  where  $k$  is a positive real number greater than 1.

Substituting this  $r_2$  into inequality relation (3-10)

$$a_3^2 - a_2 > ((k^2 + 2k + 4) \xi^2 - 1) \omega_n^2 \quad (3-11)$$

Inequality relation (3-11) indicates that for a reasonable number  $k$ , if

$\xi$  is not too small, then  $a_3^2 - a_2$  is greater than 0 and  $a_{2t} < 1.0$ .

For example if  $k=2.5$  then  $a_3^2 - a_2 > 0$  for  $\xi > .25$  and if  $k=3$  then  $a_3^2 - a_2 > 0$  for  $\xi > .229$

Thus for a fourth order system, if it has no more than one pair of complex conjugate roots of the characteristic equation and if it is adequately damped, then the coefficient  $a_{2t}$  is less than 1.

For Case 3, which is the case of two pairs of complex conjugate roots present in the characteristic equation, it is hard to set up an upper bound of  $a_{2t}$  as in Case 1 and 2. This can be seen as follows:

For Case 3

Let  $r_1 = \bar{r}_2$  ,  $r_3 = \bar{r}_4$

and  $r_1 = \xi_1 \omega_{n1} - j \sqrt{1 - \xi_1^2} \omega_{n1}$  ,  $r_3 = \xi_2 \omega_{n2} - j \sqrt{1 - \xi_2^2} \omega_{n2}$

then  $a_3^2 = (2 \xi_1 \omega_{n1} + 2 \xi_2 \omega_{n2})^2$

$$a_2 = \omega_{n1}^2 + 4 \xi_1 \xi_2 \omega_{n1} \omega_{n2} + \omega_{n2}^2$$

and  $\frac{a_2}{a_3^2} = \frac{\omega_{n1}^2 + 4 \xi_1 \xi_2 \omega_{n1} \omega_{n2} + \omega_{n2}^2}{4 \xi_1^2 \omega_{n1}^2 + 8 \xi_1 \xi_2 \omega_{n1} \omega_{n2} + 4 \xi_2^2 \omega_{n2}^2}$

Let  $\omega_{n1} < \omega_{n2}$  and  $R = \omega_{n2} / \omega_{n1}$

then  $\frac{a_2}{a_3^2} = \frac{1 + 4 \xi_1 \xi_2 R + R^2}{4 \xi_1^2 + 8 \xi_1 \xi_2 R + 4 \xi_2^2 R^2}$



Thus the ratio  $a_2/a_3^2$  depends upon  $\xi_1$ ,  $\xi_2$  and  $k$ . If both  $\xi_1$  and  $\xi_2$  are small, then  $a_2/a_3^2$  assumes a large number. However, if a well damped system is assumed, then Case 3 can be approximated to Case 2 and  $a_{2t} < 1$  is a reasonable upper bound. For example if  $\xi_1 = .3$ ,  $\xi_2 = .7$  and  $\omega_1 = \omega_2$  then  $\frac{a_2}{a_3^2} = \frac{2.84}{4.0} = .71$

In this section, in studying fourth order system curves,  $a_{2t} < 1$  is assumed. It must be noted that  $a_{2t} < 1$  is not the sufficient condition for a fourth order system to be well damped. Even though  $a_{2t}$  is very small, a fourth order system may be unstable depending upon the location of the M point. It turns out that one can find a necessary condition on  $a_{2t}$  for a system to have a relative stability greater than  $\xi_1$  a given system damping coefficient. This will be discussed later.



### 3-2. Three dimensional representation of fourth order curves.

The  $B_0$  vs  $B_1$  curves for fourth order systems defined by Equation (3-4) depend upon  $a_{2t}$  which is the coefficient of the  $s^2$  term in the characteristic equation of the transformed system equation.  $a_{2t}$  can assume any value within its allowable range and independent of frequency on the  $B_0$  vs  $B_1$  plane.

If a new variable  $B_{2t} = a_{2t}$  is defined and if  $B_{2t}$  varies on  $B_{2t}$  axis which is chosen to be perpendicular to both  $B_{1t}$  and  $B_{0t}$  axes to form a three-dimensional cartesian coordinate system, the equations

$$\begin{aligned} B_{2t} &= a_{2t} \\ B_{1t} &= a_{2t} \phi_2(s) \omega_n + \phi_3(s) \omega_n^2 + \phi_4(s) \omega_n^3 \\ B_{0t} &= a_{2t} \omega_n^2 - \phi_2(s) \omega_n^3 - \phi_3(s) \omega_n^4 \end{aligned} \quad (3-12)$$

define the Mitrovic's surfaces on the  $B_{2t} - B_{1t} - B_{0t}$  coordinate system.

For fourth order systems, Equation (3-12) defines fixed surfaces which can be applied for any fourth order system as the case of the third order chart for third order systems.

Since the variable  $B_{2t}$  is continuous and  $B_{0t}$  vs  $B_{1t}$  curves are continuous, the Mitrovic's surfaces are also continuous. Noting that any plane  $B_{2t} = a_{2t}' = \text{constant}$  which cuts the surfaces gives the set of  $B_{0t}$  vs  $B_{1t}$  curves for various values of  $\xi$ , by applying the continuous property of the Mitrovic's surfaces, all of the basic property of the  $B_0$  vs  $B_1$  curves and the analysis and design principles of feedback control systems on the  $B_0$  vs  $B_1$  plane can be extended to the three-dimensional Mitrovic's surfaces.

For example, the stable region for the M point on the  $B_0$  vs  $B_1$  plane which is the region in the first quadrant enclosed by the  $\Gamma_0$  curve can be



extended to the three-dimensional Mitrovic's surface as the region (volume) in the first octant enclosed by the  $\Gamma_0$  surface which is the surface made up by the  $\Gamma_0$  curves for all values of  $a_{2t} > 0$ . Locating the point  $M(a_{1t}, a_{0t})$  on  $B_{0t}$  vs  $B_{1t}$  plane is to locate the point  $M(a_{2t}, a_{1t}, a_{0t})$  in the Mitrovic's volume.

Fig. 3-1 is a schematic sketch of the  $\Gamma_0$  surface.  $B_{2t}$  axis is chosen so that  $B_{2t}$ ,  $B_{1t}$  and  $B_{0t}$  axes form a right-handed coordinate system. The equations defining the  $\Gamma_0$  surface are

$$\begin{aligned} B_{2t} &= a_{2t} \\ B_{1t} &= \omega_{nt}^2 \\ B_{0t} &= a_{2t}\omega_{nt}^2 - \omega_{nt}^4 \end{aligned} \quad (3-13)$$

which are the equations for a parabolic surface. The surface forms a cave with closed end at the origin on the  $B_{0t} = 0$  plane. Applying the stability criterion to each  $B_{0t}$  vs  $B_{1t}$  curve which is formed by cutting the surface with each  $B_{2t} = a_{2t}$  plane, the stable region is the whole volume inside the cave.

If the surface is cut by a plane  $B_{0t} = a_{0t} = \text{constant}$ , then there results a  $B_{1t}$  vs  $B_{2t}$  (or  $B_{2t}$  vs  $B_{1t}$ ) curve for  $\zeta = 0$  on the plane and likewise for the  $B_{0t}$  vs  $B_{2t}$  curve.

It is interesting to note that,  $B_{0t}$  vs  $B_{2t}$  curves for  $\zeta = 0$  on  $B_{0t}$  vs  $B_{2t}$  planes for fourth order systems are straight lines whose slopes and axis intersections are determined by  $a_{1t}$  only and an extremely simple method to test the absolute stability of a fourth order system can be derived.

By eliminating  $\omega_{nt}$  and  $a_{2t}$  in Equation (3-13), the  $\Gamma_0$  surface of a fourth order system is

$$B_{0t} = B_{1t} B_{2t} - B_{1t}^2 \quad (3-14)$$



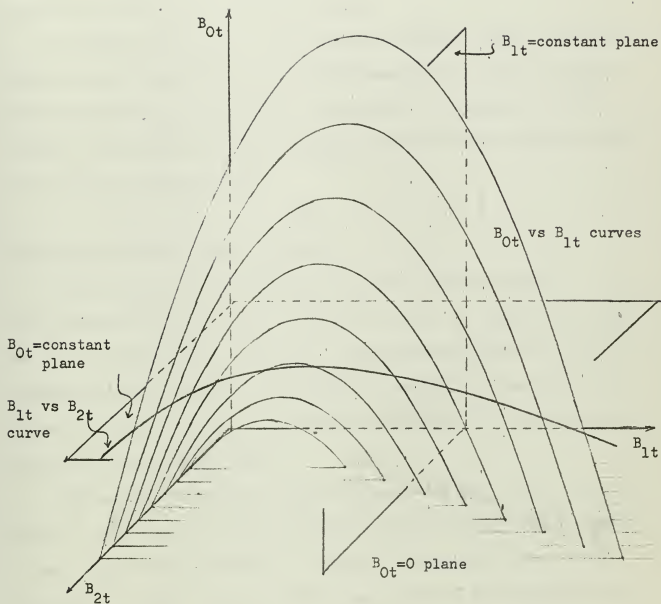


Fig. 3-1 The  $\Gamma_0$  surface for 4th order systems.



By cutting the surface represented by Equation (3-14) with a  $B_{1t} = a_{1t}$  plane, the  $B_{0t}$  vs  $B_{2t}$  curve for  $\xi = 0$  is obtained as

$$B_{0t} = a_{1t} B_{2t} - a_{1t}^2 \quad (3-15)$$

which is an equation defining a family of straight lines on the  $B_{0t}$  vs  $B_{2t}$  plane (or  $B_{1t} = a_{1t} = \text{constant}$  plane).

For each value of  $a_{1t}$ , there results a different line. By reasoning with the stable region in the three-dimensional volume, the region for the point M ( $a_{2t}$ ,  $a_{0t}$ ) on  $B_{0t}$  vs  $B_{2t}$  plane to guarantee a stable system is the area in the first quadrant below the line defined by Equation (3-15) or mathematically

$$0 < a_{0t} < a_{1t} \cdot a_{2t} - a_{1t}^2$$

which agrees with the Routh criterion.

The  $\Gamma_0$  curves for some values of  $a_{1t}$  are plotted in Fig. 3-2. As the Equation (3-15) indicates, a  $\Gamma_0$  line for  $a_{1t} = k$  cuts the  $B_{2t}$  axis at  $B_{2t} = k$  and it has a slope of  $k$ . The frequency on these  $\Gamma_0$  lines is constant at all points on a line which is a peculiar nature of the  $\Gamma_0$  curve on a  $B_{0t}$  vs  $B_{2t}$  plane. Thus the  $\Gamma_0$  line itself is a constant  $\omega_{nt}$  curve on the  $B_{0t}$  vs  $B_{2t}$  plane.

For instance, along the  $\Gamma_0$  line for  $a_{1t} = .2$ ,  $\omega_{nt} = .446$  which is equal to  $\sqrt{a_{1t}}$  as the second of Equation (3-13) implies. Thus a point M ( $a_{0t}$ ,  $a_{2t}$ ) located anywhere on a single  $\Gamma_0$  line, say for  $a_{1t} = a_{1t}'$ , represents a closed loop system which has a pair of pure imaginary roots  $r = \pm j\omega_t = \pm j' \sqrt{a_{1t}'}$  which is readily seen from the basic nature of the  $\Gamma_0$  curve.

As an example, suppose a characteristic equation of a fourth order system is

$$f(s) = s^4 + s^3 + a_{2t}s^2 + .2s + a_{0t} = 0$$







Since  $a_{1t} = .2$  the  $\Gamma_0$  line is the line labeled as  $a_{1t} = .2$ ,  $\omega_t = .446$  in Fig. 3-2. The Mitrovic's working point M ( $a_{0t}, a_{2t}$ ) located anywhere above the  $\Gamma_0$  line represents an unstable system, and dually for below the line. If the M point is followed along the line then resulting every characteristic equation has a pair of imaginary roots  $r = \pm j.446$ . For instance, if M (.7, .1) is chosen, then the characteristic equation is

$$f_1(s) = s^4 + s^3 + .7s^2 + .2s + .1 = 0$$

and  $f_1(j.446) = 0$

The point M (.7, .1) is also on the  $\Gamma_0$  line for  $a_{1t} = .5$ , therefore another characteristic equation

$$f_2(s) = s^4 + s^3 + .7s^2 + .5s + .1 = 0$$

has the roots  $r = \pm j\sqrt{5}$  and it calculates to

$$f_2(j\sqrt{5}) = 0$$

From the Mitrovic's basic equation and the way of defining a three-dimensional plot, the concept of a Mitrovic's surface can be generalized to any order system and to a set of any three coefficients of the characteristic equation. Then fixing one of the three axes to a constant, there results a well-defined curve which is the one defined by the generalized Mitrovic's equation in Reference 3, though the generalized Mitrovic's equation turns out to be undefined and uninterpretable at some particular values of  $\xi$  for some particular set of coefficients. (For example, the  $\Gamma_0$  curve for the  $B_{0t}$  vs  $B_{2t}$  plot.)

The most significant feature in the concept of the Mitrovic's surface is the possibility of controlling three coefficients simultaneously since the surface is independent of the three coefficients which are to be controlled.



### 3-3. Study of fourth order curves.

As pointed out in Section 3-1, a fourth order system can be transformed into an equivalent system so that there remains only one parameter (coefficient) in the characteristic equation on which the fourth order system curves depend. This feature greatly simplifies studying the behavior of the Mitrovic's curves for fourth order systems.

In Section 3-2, it was shown that if a fourth order system characteristic equation is transformed such that the coefficients of the  $S^4$  and  $S^3$  terms are unity, then the coefficient of the  $S^2$  term  $a_{2t}$  which is the only one affecting  $B_{0t}$  vs  $B_{1t}$  curves is less than 1.0 for most practical fourth order systems.

Since the  $B_{0t}$  vs  $B_{1t}$  curves are continuous for all values of  $\omega_{nt}$ ,  $\zeta$ , and the coefficients of characteristic equation, if the behavior of the curves is well investigated and if the curves for some pertinent values of parameters are constructed, then those preconstructed curves may serve as a guide or charts for both analysis and design of fourth order systems

In this section, the effect of  $a_{2t}$  on  $B_{0t}$  vs  $B_{1t}$  curves is studied to provide the fourth order charts.

In Equation (3-4) which is the equation defining  $B_{0t}$  vs  $B_{1t}$  curves for a fourth order system, there are two parameters  $a_{2t}$  and  $\zeta$ . For one value of  $a_{2t}$ , each  $\zeta$  gives a different curve  $\Gamma_{\zeta}$  to form a family of curves corresponding to the particular value of  $a_{2t}$ . This family is denoted by  $\{\zeta\}_{a_{2t}}$ , or simply  $\{\zeta\}$ . Then, for each  $a_{2t}$  there results a different family.

Fig. 3-3 to Fig. 3-22 show such families. On each sheet the curves for five values of  $\zeta$  are shown. For one value of  $a_{2t}$ , there are



040

 $B_{0t}$ 

035

030

Fig. 3-3  $B_{0t}$  vs  $B_{1t}$  curves (4th order systems)for  $a_{2t}=.1$ ; ( $\zeta=0.,.1,.2,.3,.4,.5$ )

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020

015

010

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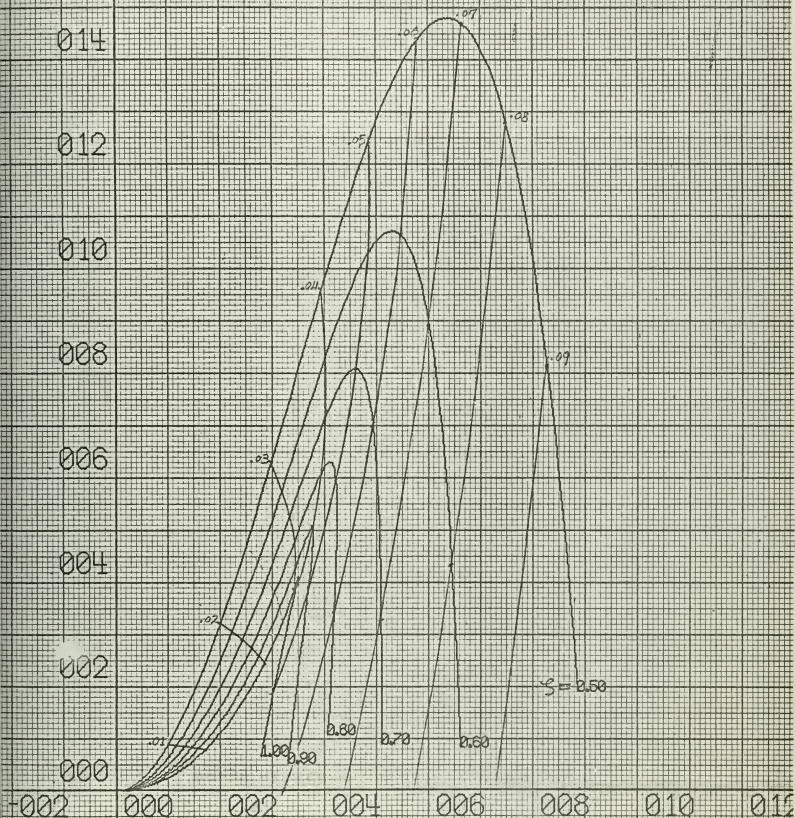
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-005

-002 000 002 004 006 008 010 012  $B_{1t}$ X AXIS SCALE =  $2.00E-02$ Y AXIS SCALE =  $5.00E-04$



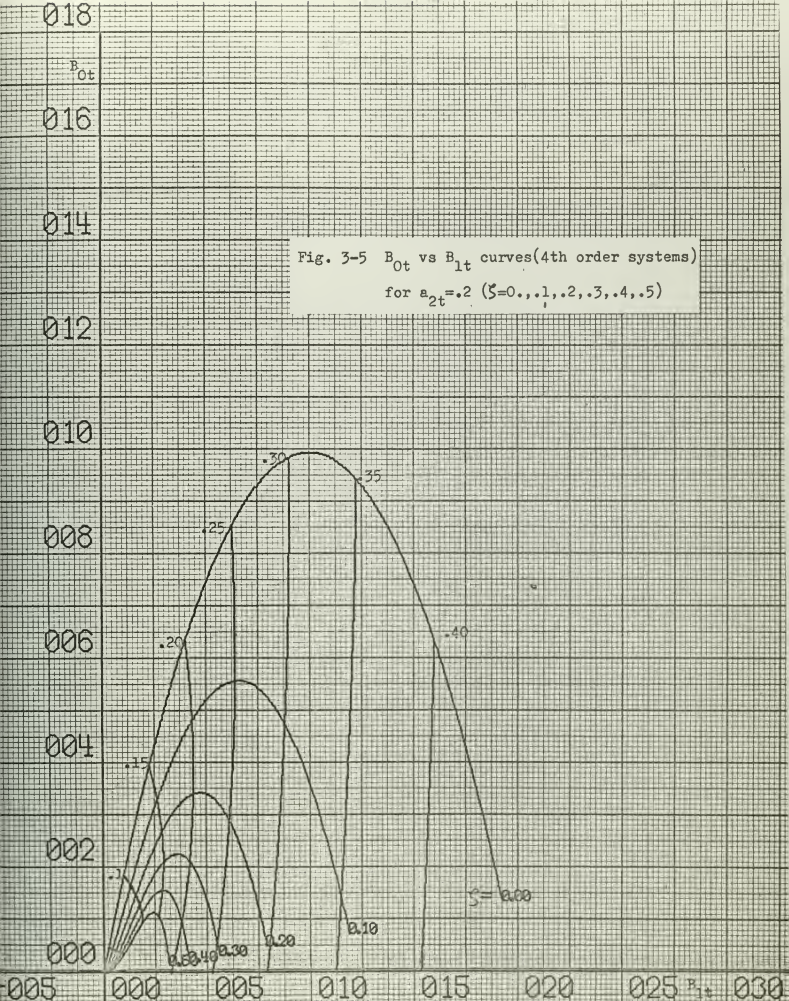
Fig. 3-4  $B_{0t}$  vs  $B_{1t}$  curves (4th order systems)  
for  $a_{2t} = .1$  ( $\xi = .5, .6, .7, .8, .9, 1.0$ )



X AXIS SCALE =  $2.00E-03$

Y AXIS SCALE =  $2.00E-05$





X AXIS SCALE =  $5.00E-02$   
Y AXIS SCALE =  $2.00E-03$



$B_{0t}$   
016

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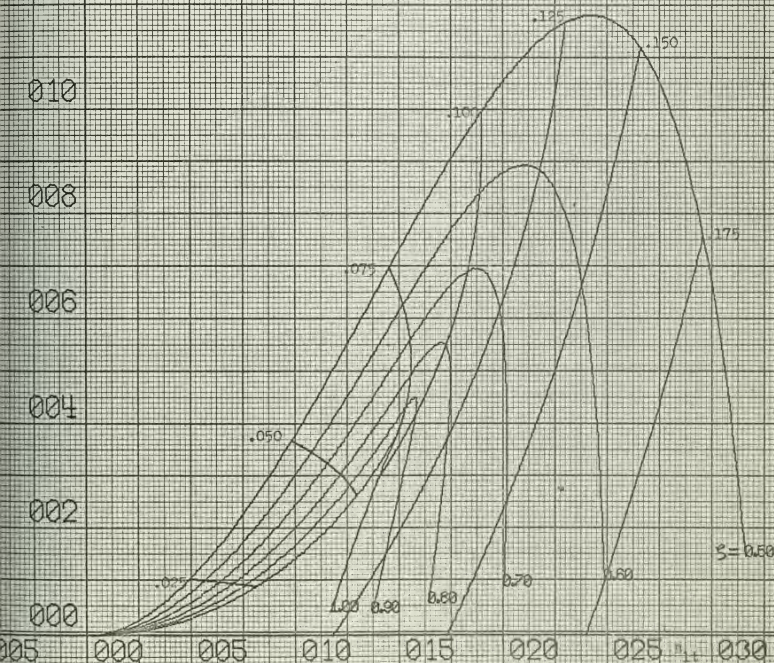
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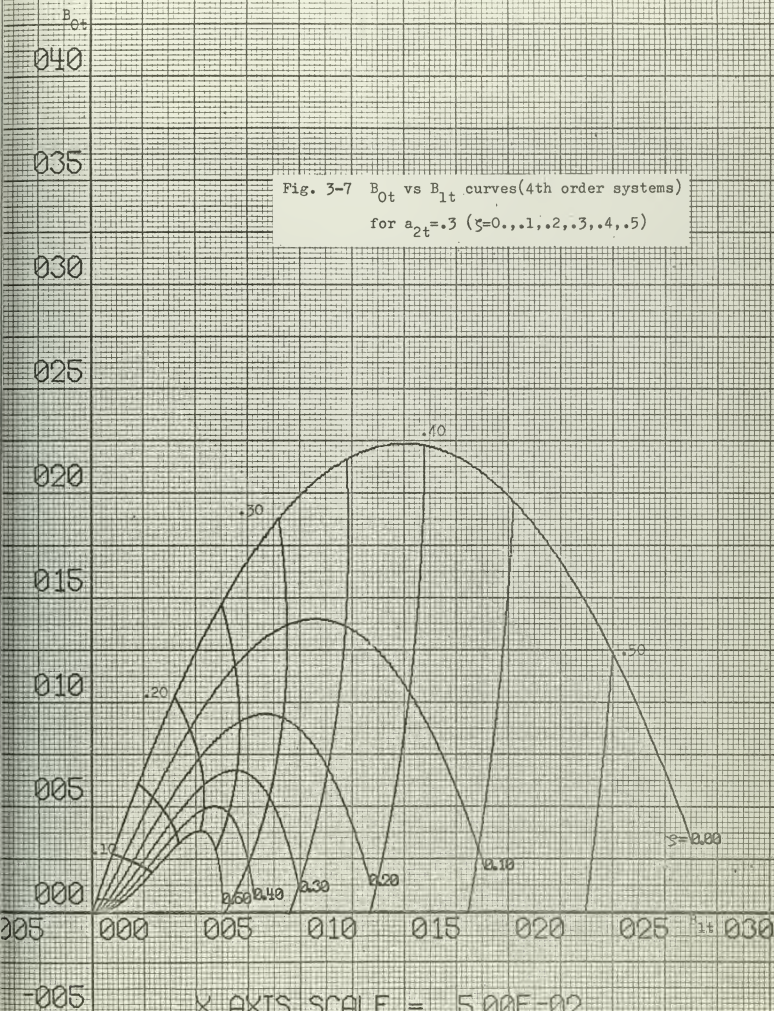
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Fig. 3-6  $B_{0t}$  vs  $B_{1t}$  curves(4th order systems)  
for  $a_{2t}=.2$  (  $=.5,.6,.7,.8,.9,1.0$  )



X AXIS SCALE =  $5.00E-03$   
Y AXIS SCALE =  $2.00E-04$

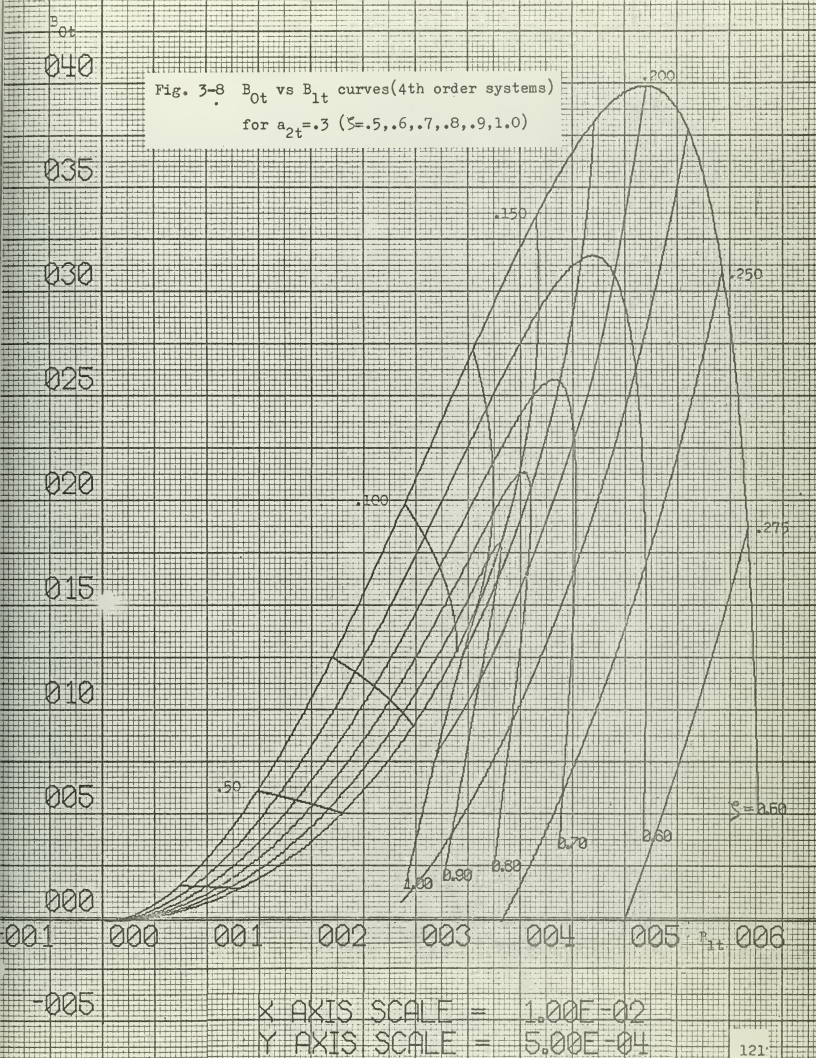




X AXIS SCALE =  $5.00E-02$

Y AXIS SCALE =  $5.00E-03$







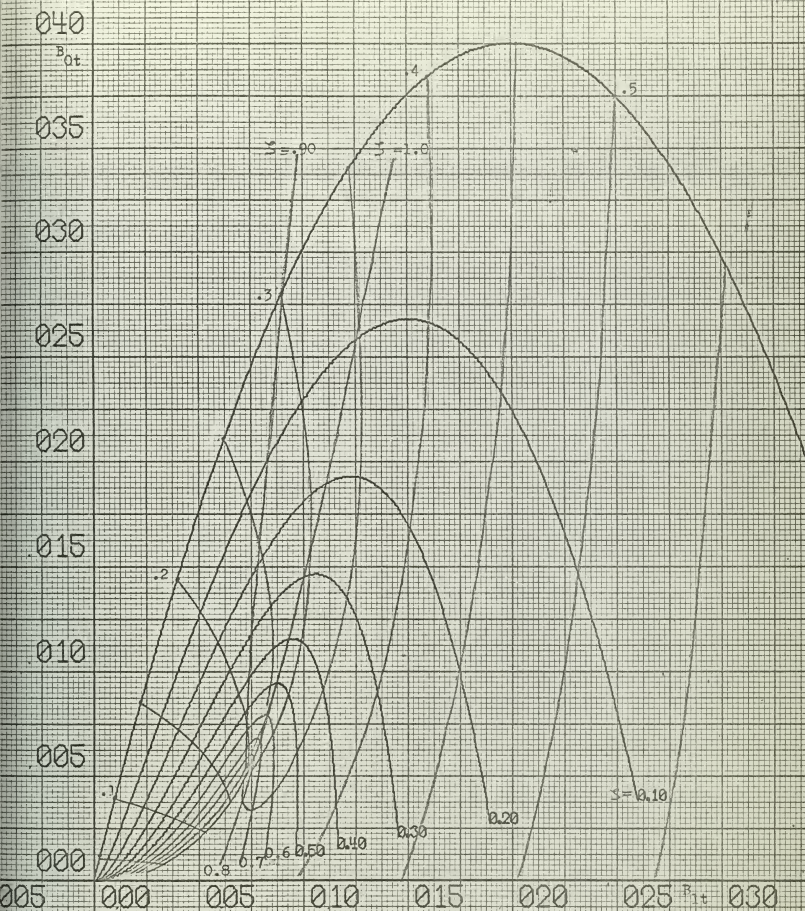


Fig. 3-9  $B_{0t}$  vs  $B_{1t}$  curves (4th order systems)

for  $a_{2t} = .4$  ( $\zeta = 0., .1, .2, .3, .4, .5, .6, .7, .8, .9, 1.0$ )

X AXIS SCALE =  $5.00E-02$

Y AXIS SCALE =  $5.00E-03$



016

 $B_{0t}$ 

014

012

Fig. 3-10  $B_{0t}$  vs  $B_{1t}$  curves (4th order systems)  
for  $a_{2t} = .4$  ( $\xi = .5, .6, .7, .8, .9, 1.0$ )

010

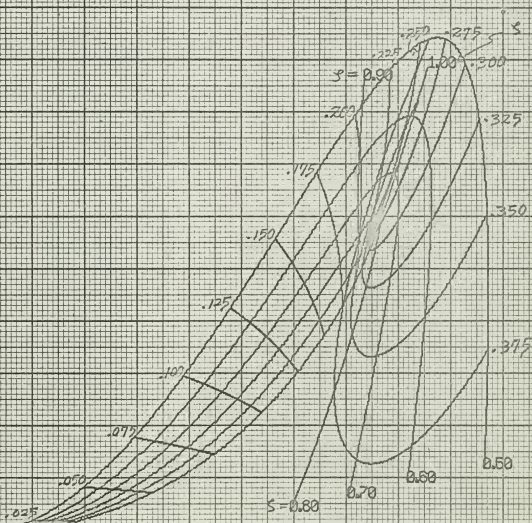
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 $B_{1t}$ 

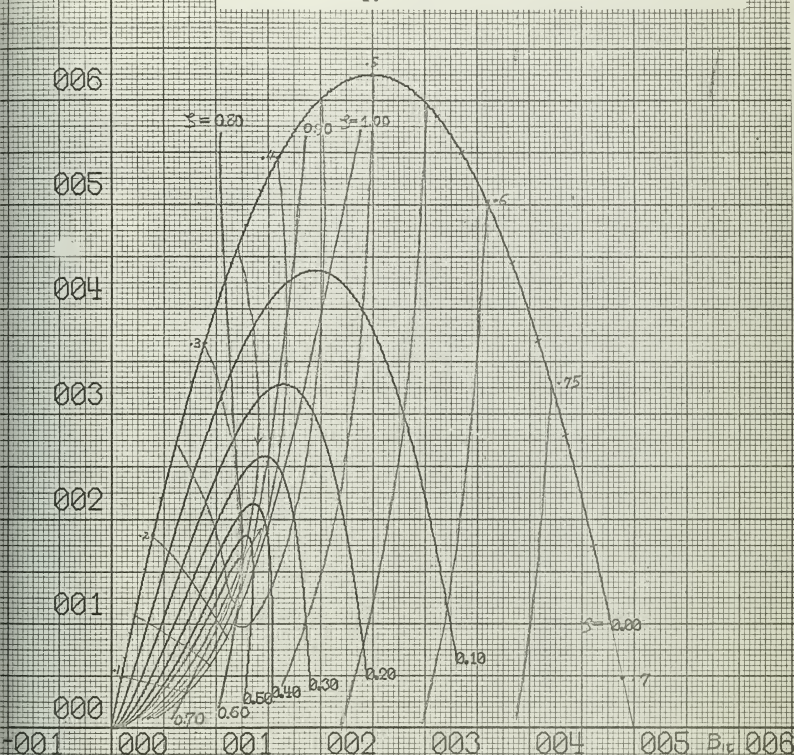
012

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X AXIS SCALE =  $2.00E-02$ Y AXIS SCALE =  $2.00E-03$



Fig. 3-11  $B_{0t}$  vs  $B_{1t}$  curves (4th order systems)  
for  $a_{2t} = .5$  ( $\xi = 0., .1, .2, .3, .4, .5, .6, .7, .8, .9, 1.0$ )

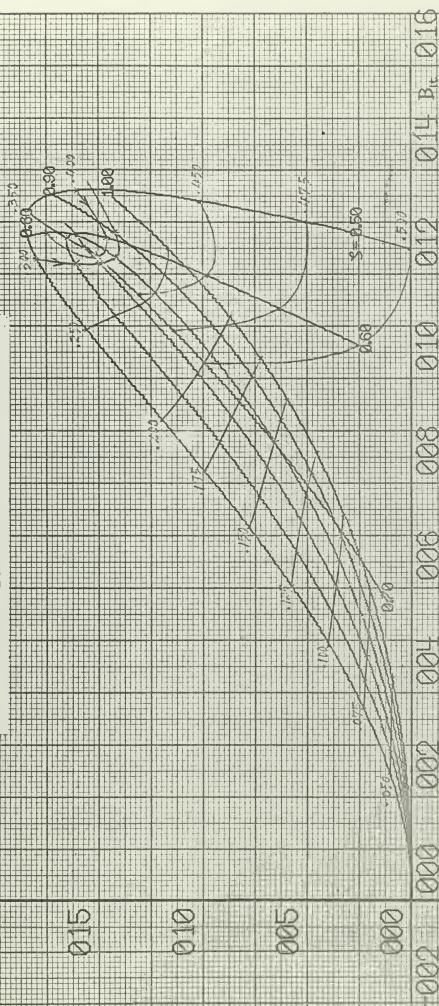


X AXIS SCALE = 1.00E-01

Y AXIS SCALE = 1.00E-02



Fig. 3-12  $P_{ot}$  vs  $P_{lt}$  curves (4th order systems)  
for  $a_t = 5$  ( $\zeta = 5, 6, 7, 8, 9, 1.0$ )

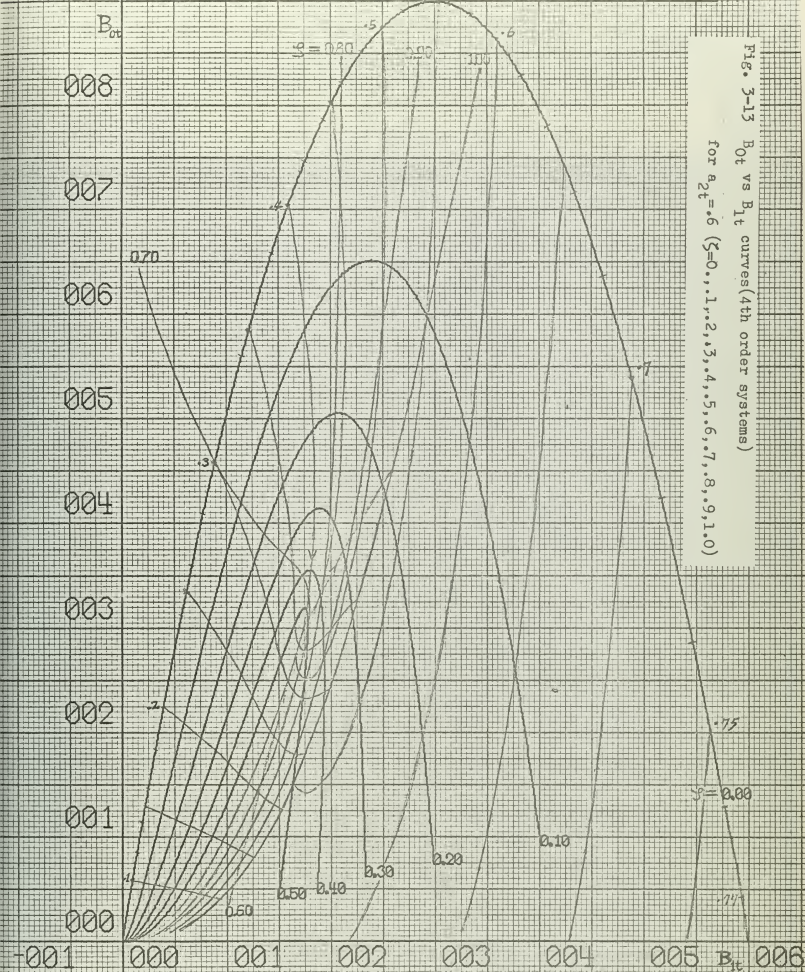


X AXIS SCALE = 2.00E-02  
Y AXIS SCALE = 5.00E-03



Fig. 3-13  $B_{0t}$  vs  $B_{1t}$  curves (4th order systems)

for  $a_{2t} = .6$  ( $\zeta = 0, .1, .2, .3, .4, .5, .6, .7, .8, .9, 1.0$ )



X AXIS SCALE =  $1.00E-01$

Y AXIS SCALE =  $1.00E-02$



040

 $B_{0t}$ 

035

Fig. 3-14  $B_{0t}$  vs  $B_{1t}$  curves (4th order systems)for  $a_{2t} = .6$  ( $\zeta = .5, .6, .7, .8, .9, 1.0$ )

030

025

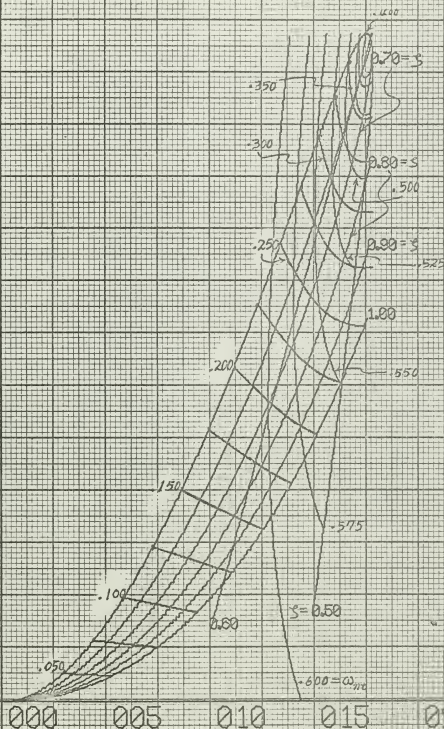
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000

X AXIS SCALE =  $5.00E-02$ Y AXIS SCALE =  $5.00E-03$



016

 $B_{0t}$ 

014

Fig. 3-15  $B_{0t}$  vs  $B_{1t}$  curves (4th order systems)for  $a_{2t} = .7$  ( $\xi = 0., .1, .2, .3, .4, .5, .6, .7, .8, .9, 1.0$ )

012

010

008

006

004

002

000

01 000 001 002 003 004 005 006  $B_{1t}$ 

-002

AXIS SCALE = 1.00E-01

OXS SCALE = 2.00E-02



008

Bot

007

Fig. 3-16  $B_{0t}$  vs  $B_{1t}$  curves (4th order systems)  
for  $a_{2t} = .7$  ( $\zeta = .5, .6, .7, .8, .9, 1.0$ )

006

005

004

003

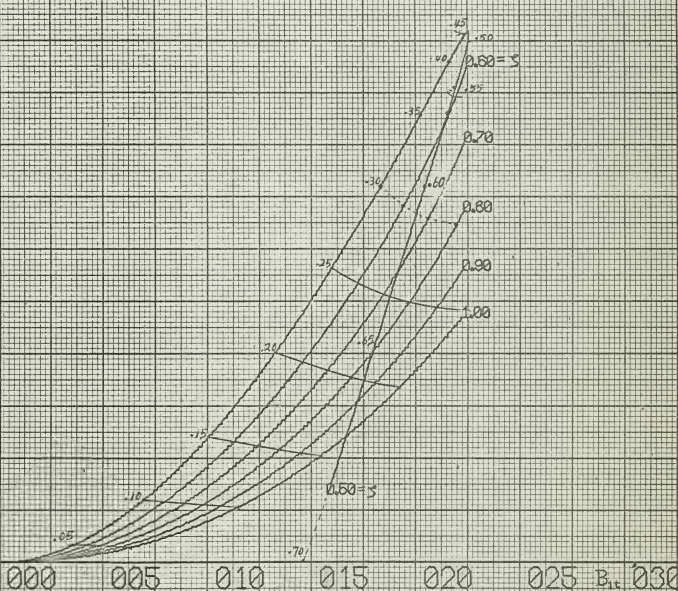
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-005	000	005	010	015	020	025	B <sub>tt</sub>	030
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-001



X AXIS SCALE = 5.00E-02

Y AXIS SCALE = .100E-02



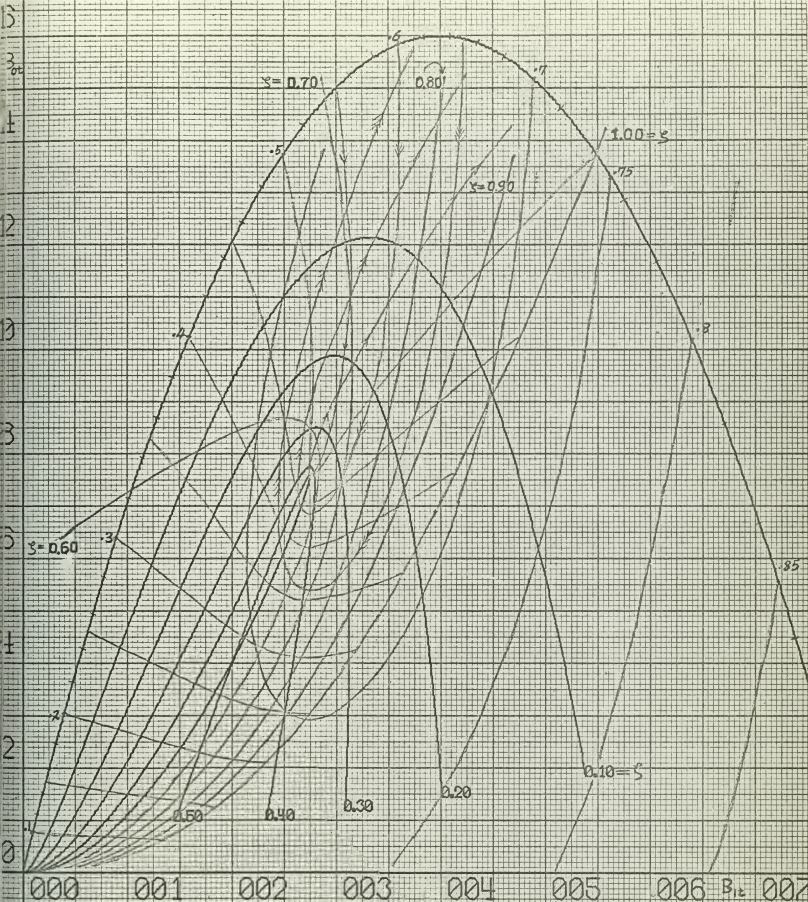


Fig. 3-17  $B_{0t}$  vs  $B_{1t}$  curves (4th order systems)

for  $a_{2t} = .8$  ( $\zeta = 0., .1., .2., .3., .4., .5., .6., .7., .8., .9., 1.0$ )

SCALE = 1.00E-01

SCALE = 2.00E-02



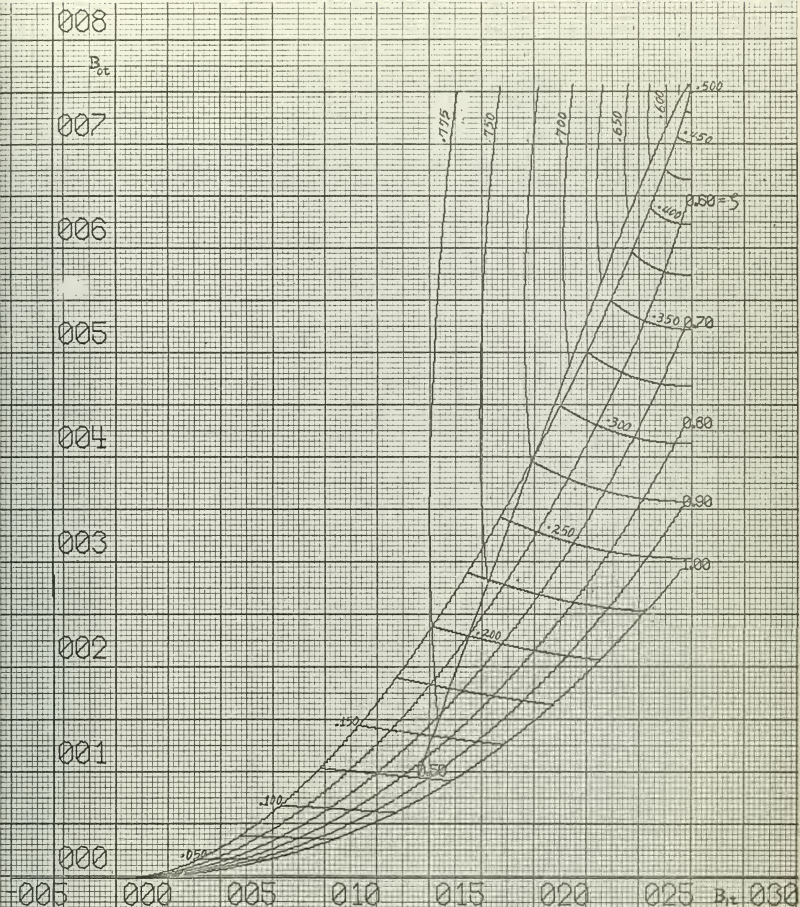


Fig. 3-18  $B_{0t}$  vs  $B_{1t}$  curves(4th order systems)

for  $a_{2t} = .8$  ( $\zeta = .5, .6, .7, .8, .9, 1.0$ )

Y AXIS SCALE =  $5.00E-02$

Y AXIS SCALE =  $1.00E-02$



Fig. 3-19  $B_{Ot}$  vs  $B_{lt}$  curves (4th order systems)  
for  $a_{2t} = .9$  ( $\zeta = 0, .1, .2, .3, .4, .5, .6, .7, .8, .9, 1.0$ )





016

 $B_{0t}$ 

014

012

010

008

006

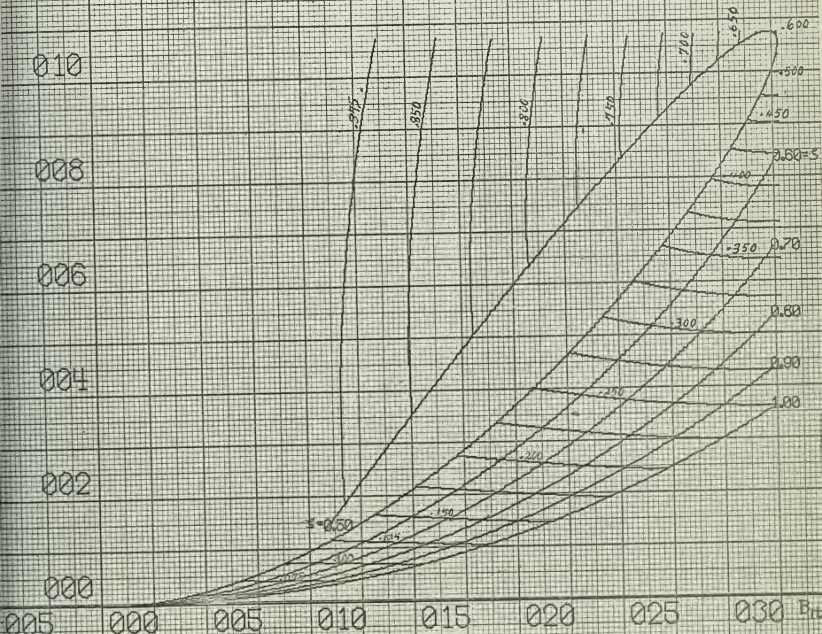
004

002

000

-002

Fig. 3-20  $B_{0t}$  vs  $B_{1t}$  curves (4th order systems)  
for  $a_{2t} = .9$  ( = .5, .6, .7, .8, .9, 1.0 )



X AXIS SCALE = 5.00E-02

Y AXIS SCALE = 2.00E-02



040

 $B_{ot}$ 

035

030

Fig. 3-21  $B_{ot}$  vs  $B_{lt}$  curves (4th order systems)  
for  $a_{2t}=1.0$  ( $\zeta=0, .1, .2, .3, .4, .5, 1.0$ )

025

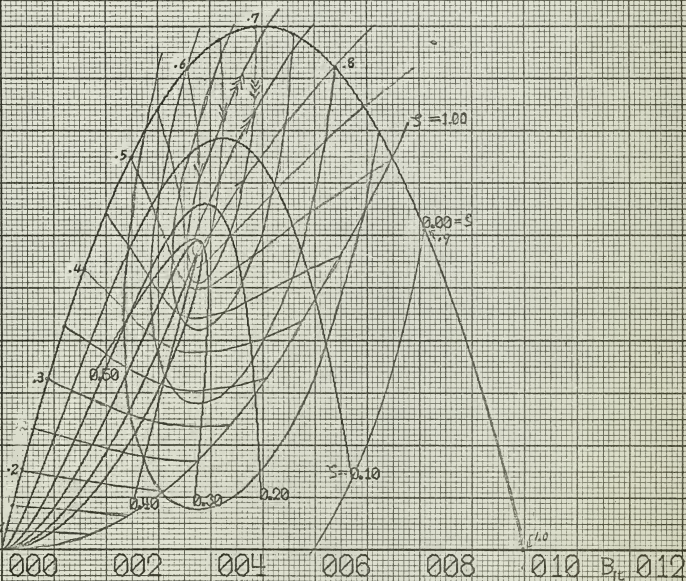
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X AXIS SCALE =  $2.00E-01$ Y AXIS SCALE =  $5.00E-02$



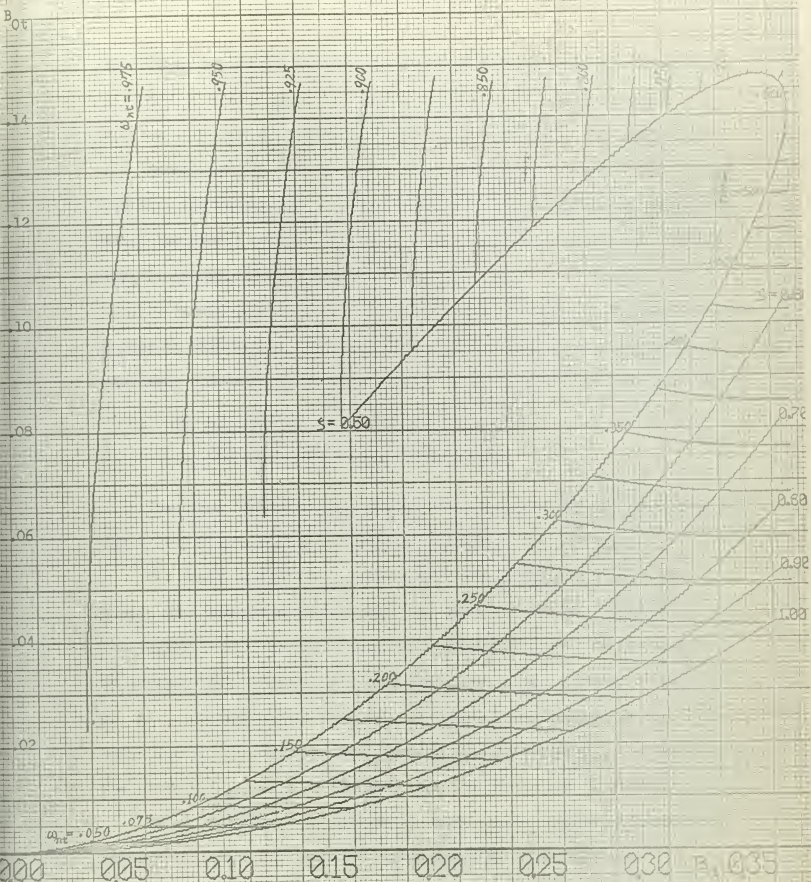


Fig. 3-22  $B_{0t}$  vs  $B_{1t}$  curves (4th order systems)

for  $a_{2t} = 1.0$  ( $\zeta = .5, .6, .7, .8, .9, 1.0$ )



associated two sheets of graphs, one for  $0 \leq \zeta \leq .5$  and the other for  $.5 \leq \zeta \leq 1.0$  with interval of  $\zeta$  .1. The values of  $a_{2t}$  are from .1 to 1.0, thus for every two sheets of graphs,  $a_{2t}$  increases by interval of .1. (Note that for  $.4 \leq a_{2t} \leq 1.0$ , the first graph for each value of  $a_{2t}$  contains the family  $\{\zeta\}$  for  $0 \leq \zeta \leq 1.0$  and the second graph contains the family  $\{\zeta\}$  for  $.5 \leq \zeta \leq 1.0$ .) The constant  $\omega_{nt}$  curves are also shown for some pertinent values of  $\omega_{nt}$ .

It is observed that as  $a_{2t}$  increases, the curves are in general expanded in both abscissa and ordinate. This is well expected from the equations defining  $B_{0t}$  vs  $B_{1t}$  curves. The equations are rewritten below

$$\begin{aligned} B_{1t} &= a_{2t} \phi_2(\zeta) \omega_{nt}^2 + \phi_3(\zeta) \omega_{nt}^3 + \phi_4(\zeta) \omega_{nt}^4 \\ B_{0t} &= a_{2t} \omega_{nt}^2 - \phi_2(\zeta) \omega_{nt}^3 - \phi_3(\zeta) \omega_{nt}^4 \end{aligned} \quad (3-4)$$

The terms with which  $a_{2t}$  contributes to  $B_{1t}$  and  $B_{0t}$  are non-negative for  $0 \leq \zeta \leq 1.0$ , thus increasing  $a_{2t}$  increases  $B_{1t}$  and  $B_{0t}$ .

For small values of  $a_{2t}$  each member of the family  $\{\zeta\}$  is so shaped that the whole region under the curve in the first quadrant is the region for the M point where the relative stability is guaranteed to be greater than the value of  $\zeta$  of the curve. Such a region is denoted by  $R_\zeta$ . For example in Fig. 3-3, the whole area enclosed by the  $\zeta_{.5}$  curve and  $B_{1t}$  axis in the first quadrant is the region  $R_{.5}$  on which if the point M ( $a_{1t}$ ,  $a_{0t}$ ) is located, then the system has  $\zeta$  greater than .5.

As  $a_{2t}$  increases, the manner in which the  $R_\zeta$  region is formed is changed in such a way that the curves for high  $\zeta$  do not form  $R_\zeta$ .

The behavior of the curves depends upon the values of  $a_{2t}$  and  $\zeta$ . Examination of Equation (3-4) reveals that for a particular value of  $a_{2t}$ , there is a certain critical value of  $\zeta$  such that if  $\zeta$  is increased



beyond the critical value, then the  $\Gamma_{\zeta}$  curve exhibits counter-clockwise encirclement on the  $B_{0t}$  vs  $B_{1t}$  plane so that there is formed a small  $R_{\zeta}$  near to the origin or no  $R_{\zeta}$  at all. Equivalently for a particular value of  $\zeta$  there exists a critical value of  $a_{2t}$ .

Investigation of the critical value of  $\zeta$  or  $a_{2t}$  is important since in synthesis of a system one must choose a point  $M(a_{1t}, a_{0t})$  so that the specified system performances are met and particularly for the relative stability of the system, the  $M$  point must be located on a  $R_{\zeta}$  where  $\zeta$  is some suitable value to meet the specified relative stability. For example if the system relative stability is specified by  $\zeta = .5$  then one should not choose the  $M$  point anywhere on the  $\Gamma_{.5}$  curve for  $a_{2t} = 1.0$ . (See Fig. 3-21 and 3-22.)

In synthesis of a system, usually  $\zeta$  is chosen at some pertinent value according to the required relative stability. Therefore it is convenient to investigate the critical value of  $a_{2t}$  for given values of  $\zeta$ , thus a study of the behavior of the curves with changing  $a_{2t}$  for a fixed  $\zeta$  is needed. This requires that the family of curves must be represented in a different fashion from those of Fig. 3-3 through Fig. 3-22 in which the families of  $B_{0t}$  vs  $B_{1t}$  curves with changing  $\zeta$  for fixed  $a_{0t}$  (the family  $\{\zeta, a_{2t}\}$ ) are represented. That is, the family of curves must be represented with changing  $a_{2t}$  for fixed  $\zeta$ . Those families are denoted by  $\{\zeta, \{a_{2t}\}\}$ .

At this point, in connection with the study of the families  $\{\zeta, \{a_{2t}\}\}$ , a constant  $\zeta$  plane is defined as follows:

A constant  $\zeta$  plane is a Mitrovic's plane on which Mitrovic's curves for fixed  $\zeta$  are plotted, and denoted by the  $\zeta = k$  plane where  $k$  is a



constant and  $0 \leq k \leq 1.0$ . With this definition, for the fourth order systems, the family of  $B_{0t}$  vs  $B_{1t}$  curves with various values of  $a_{2t}$  for  $\xi$  fixed to zero consists of all of the  $B_{0t}$  vs  $B_{1t}$  curves for  $\xi = 0$  on the  $\xi = 0$  plane, and likewise for any other values of  $0 \leq \xi \leq 1.0$ .

The constant  $\omega_n$  lines on the constant  $\xi$  plane

In general, if the Mitrovics curves are plotted on a constant  $\xi$  plane, and if any one of the coefficients which appears in the equations defining the Mitrovic's curves is varied, then all of the points at which the frequencies are the same are on a single straight line whose slope and the coordinate axis intersections are determined by  $\xi$  and  $\omega_n$ . This line is called the constant  $\omega_n$  line. (Similarly the constant  $\omega_{nt}$  line for a transformed system.)

For  $B_0$  vs  $B_1$  curves defined in general form as in Equation (2-2) by fixing  $\xi = \xi_1$ , a constant, the equation of  $B_0$  vs  $B_1$  curves on the  $\xi_1$  plane are:

$$B_1 = \sum_{k=2}^n a_k \phi_k(\xi_1) \omega_n^{k-1}$$

$$B_0 = \sum_{k=2}^n -a_k \phi_{k-1}(\xi_1) \omega_n^k$$
(3-16)

If the coefficients  $a_k$ 's,  $k \neq 1$  are fixed and if  $a_1$  is varied, then on the  $\xi_1$  plane, a family of  $B_0$  vs  $B_1$  curves  $\overrightarrow{\xi_1 \{a_i\}}$  is formed.

Rewriting Equation (3-16) to separate the term containing the variable coefficient  $a_1$  from the terms of the fixed coefficients,



$$B_1 = a_i \phi_i(\zeta_1) \omega_n^{i-1} + \sum_{\substack{k=2 \\ k \neq i}}^n a_k \phi_k(\zeta_1) \omega_n^{k-1} \quad (3-17)$$

$$B_0 = -a_i \phi_{i-1}(\zeta_1) \omega_n^i - \sum_{\substack{k=2 \\ k \neq i}}^n a_k \phi_{k-1}(\zeta_1) \omega_n^k$$

Suppose any two members of the family  $\sqrt{\zeta_1, \{a_i\}}$ , one for  $a_i = a_{i1}$  and the other for  $a_i = a_{i2}$ . Let the increment in  $a_i$  be  $\Delta a_i = a_{i2} - a_{i1}$ , then the increments in the coordinates  $B_1$  and  $B_0$  at any frequency associated with  $\Delta a_i$  are

$$\begin{aligned} \Delta B_1 &= \Delta a_i \phi_i(\zeta_1) \omega_n^{i-1} \\ \Delta B_0 &= -\Delta a_i \phi_{i-1}(\zeta_1) \omega_n^i \\ \frac{\Delta B_0}{\Delta B_1} &= -\phi_{i-1}(\zeta_1) \omega_n / \phi_i(\zeta_1) \end{aligned} \quad (3-18)$$

Equation (3-18) implies that for any change  $\Delta a_i$  in the coefficient  $a_i$  the ratio of the changes in the coordinates  $\Delta B_0 / \Delta B_1$  is independent of  $a_i$  at all frequencies, and that the distance between any two points at the same frequencies one on the curve for  $a_{i1}$  and the other on the curve for  $a_{i2}$  is linearly varying with  $\Delta a_i$ .

For the family  $\sqrt{\zeta_1, \{a_i\}}$  on the  $\zeta_1$  plane, consider a point at  $\omega_n = \omega_{n1}$  on each member of the family. If  $a_i$  assumes a continuous set of values, then the locus of the point of  $\omega_n = \omega_{n1}$  on the  $\zeta_1$  plane is obtained by eliminating  $a_i$  in Equation (3-17) and replacing  $\omega_n$  by  $\omega_{n1}$ , namely

$$B_0 = -\frac{\phi_{i-1}(\zeta_1) \omega_{n1}}{\phi_i(\zeta_1)} \left( B_1 - \sum_{\substack{k=2 \\ k \neq i}}^n a_k \phi_k(\zeta_1) \omega_{n1}^{k-1} \right) - \sum_{\substack{k=2 \\ k \neq i}}^n a_k \phi_{k-1}(\zeta_1) \omega_{n1}^k \quad (3-19)$$



$$\text{or } B_0 = -\frac{\phi_{i-1}(s_1)\omega_{n1}}{\phi_i(s_1)} B_1 + \sum_{\substack{k=2 \\ k \neq i}}^n a_k \omega_{n1}^k \left( \frac{\phi_{i-1}(s_1)\phi_k(s_1) - \phi_i(s_1)\phi_{k-1}(s_1)}{\phi_i(s_1)} \right) \quad (3-19a)$$

Since  $s_1$  is fixed to  $s_1$  on the  $s_1$  plane, and  $\omega_n$  is fixed at  $\omega_{n1}$ ,

Equation (3-19) describes a line with a slope of  $-\frac{\phi_{i-1}(s_1)\omega_{n1}}{\phi_i(s_1)}$  pass-

ing through the point  $\left( \sum_{\substack{k=2 \\ k \neq i}}^n a_k \phi_k(s_1) \omega_{n1}^{k-1}, -\sum_{\substack{k=2 \\ k \neq i}}^n a_k \phi_{k-1}(s_1) \omega_{n1}^k \right)$

on the  $s_1$  plane. Thus the equation defining the constant  $\omega_n$  line for

$\omega_n = \omega_{n1}$  on the  $s_1$  plane with carrying  $a_i$  is obtained.

The axis intersections of the constant  $\omega_n$  line are found by letting

$B_1 = 0$  and  $B_0 = 0$  in Equation (3-19) or (3-19a)

$$\begin{aligned} B_1 \text{ intersect.} &= \sum_{\substack{k=2 \\ k \neq i}}^n a_k \phi_k(s_1) \omega_{n1}^{k-1} - \frac{\phi_i(s_1)}{\phi_{i-1}(s_1)} \sum_{\substack{k=2 \\ k \neq i}}^n a_k \phi_{k-1}(s_1) \omega_{n1}^{k-1} \\ &= \sum_{\substack{k=2 \\ k \neq i}}^n a_k \omega_{n1}^{k-1} \left( \frac{\phi_{i-1}(s_1)\phi_k(s_1) - \phi_i(s_1)\phi_{k-1}(s_1)}{\phi_{i-1}(s_1)} \right) \end{aligned} \quad (3-20)$$

$$B_0 \text{ intersect.} = \sum_{\substack{k=2 \\ k \neq i}}^n a_k \omega_{n1}^k \left( \frac{\phi_{i-1}(s_1)\phi_k(s_1) - \phi_i(s_1)\phi_{k-1}(s_1)}{\phi_i(s_1)} \right) \quad (3-21)$$

In particular if  $i = 2$  then noting that  $-\phi_2\phi_{k-1} - \phi_{k-2} = \phi_k$

and  $\phi_1 = -1$ , Equations (3-19), (3-20) and (3-21) reduce to



$$B_0 = \frac{\omega_{n1}}{\phi_2(\xi_1)} B_1 + \sum_{k=3}^n \bar{a}_k \omega_{n1}^k \frac{\phi_{k-2}(\xi_1)}{\phi_2(\xi_1)} \quad \text{or}$$

$$B_0 = \frac{\omega_{n1}}{\phi_2(\xi_1)} \left( B_1 + \sum_{k=3}^n \bar{a}_k \phi_{k-2}(\xi_1) \omega_{n1}^{k-1} \right) \quad (3-22)$$

$$B_{1 \text{ intersect.}} = - \sum_{k=3}^n \bar{a}_k \phi_{k-2}(\xi_1) \omega_{n1}^{k-1} \quad (3-23)$$

$$B_{0 \text{ intersect.}} = \sum_{k=3}^n \frac{\bar{a}_k \phi_{k-2}(\xi_1)}{\phi_2(\xi_1)} \omega_{n1}^k \quad (3-24)$$

For a fourth order system whose  $B_0$  vs  $B_1$  curves are defined by Equation (3-4) which is for the transformed system, the constant  $\omega_{nt}$  lines for  $\omega_{nt} = \omega_{t1}$  on the  $\xi_1$  plane with  $a_{2t}$  varying is obtained from Equation (3-19) or (3-22) by proper substitutions, thus

$$B_{0t} = \frac{\omega_{t1}}{\phi_2(\xi_1)} \left( B_{1t} - \omega_{t1}^2 (1 - \phi_2(\xi_1) \omega_{t1}) \right) \quad (3-25)$$

with axis intersections given by

$$B_{1t} \text{ intersection} = \omega_{t1}^2 (1 - \phi_2(\xi_1) \omega_{t1}) \quad (3-26)$$

$$B_{0t} \text{ intersection} = - \frac{\omega_{t1}^3}{\phi_2(\xi_1)} (1 - \phi_2(\xi_1) \omega_{t1}) \quad (3-27)$$

Equation (3-25) defines the constant  $\omega_{nt}$  lines for all values of  $\omega_{nt}$  on the  $\xi_1$  plane for fourth order systems. The slopes of the constant  $\omega_{nt}$  lines are proportional to the frequencies and inversely proportional to  $\xi_1$  ( $\phi_2(\xi) = 2\xi$ ). For  $\xi = 0$ , the slopes are infinity and all constant  $\omega_{nt}$  lines on the  $\xi = 0$  plane are vertical lines.



In view of Equations (3-26) and (3-27), if  $\omega_{t1} < \frac{1}{2}\xi_1$  then

$$B_{1t} \text{ intersection} > 0$$

$$B_{0t} \text{ intersection} < 0 \quad \text{and dually.}$$

For the maximum values of the  $B_{1t}$  intersection and minimum value of  $B_{0t}$  intersection, differentiating Equations (3-26) and (3-27) give

$$\text{Max. } B_{1t} \text{ intersect.} = 1/27\xi^2 \quad \text{at } \omega_{t1} = 1/3\xi_1$$

$$\text{Min. } B_{0t} \text{ intersect.} = -27/2048\xi_1^3 \quad \text{at } \omega_{t1} = 3/8\xi_1$$

Thus the constant  $\omega_{nt}$  line of the fourth order system moves to the right with increasing slope as  $\omega_{nt}$  is increased. Then at  $\omega_{nt} = 1/3\xi$  the maximum  $B_{1t}$  intersect. is reached. As the frequency  $\omega_{nt}$  is increased beyond  $1/3\xi$  then the constant  $\omega_{nt}$  line moves back to the left with a steadily increasing slope.

Fig. 3-23, 3-24 and 3-25 are families of  $B_{0t}$  vs  $B_{1t}$  curves  $\sqrt{\xi, a_{2t}}$  plotted on the  $\xi = 0, .1$  and  $.2$  planes respectively. The coefficient  $a_{2t}$  is varied from  $.1$  to  $1.0$  with  $\Delta a_{2t} = .1$ . The constant  $\omega_{nt}$  lines are shown some pertinent values of frequencies, and those can be compared with the constant  $\omega_{nt}$  curves in Fig. 3-3 to Fig. 3-22 which are the families  $\sqrt{\xi, a_{1t}}$ .

The property of the constant  $\omega_{nt}$  line can be used in constructing a  $B_{0t}$  vs  $B_{1t}$  curve on a constant  $\xi$  plane for any value of  $a_{2t}$  when there are on the plane any two curves each for a known value of  $a_{2t}$ . This is illustrated in Fig. 3-26.

Suppose two  $\sqrt{\xi}$  curves, one for  $a_{2t} = .30$  and the other for  $a_{2t} = .40$ , are preconstructed on the  $\xi = .5$  plane. The line segments joining the pairs of two points at the same frequency on the two curves are the constant  $\omega_{nt}$  lines. For example, the points  $P_1$  and  $P_2$  are at



040

 $B_{0t}$ 

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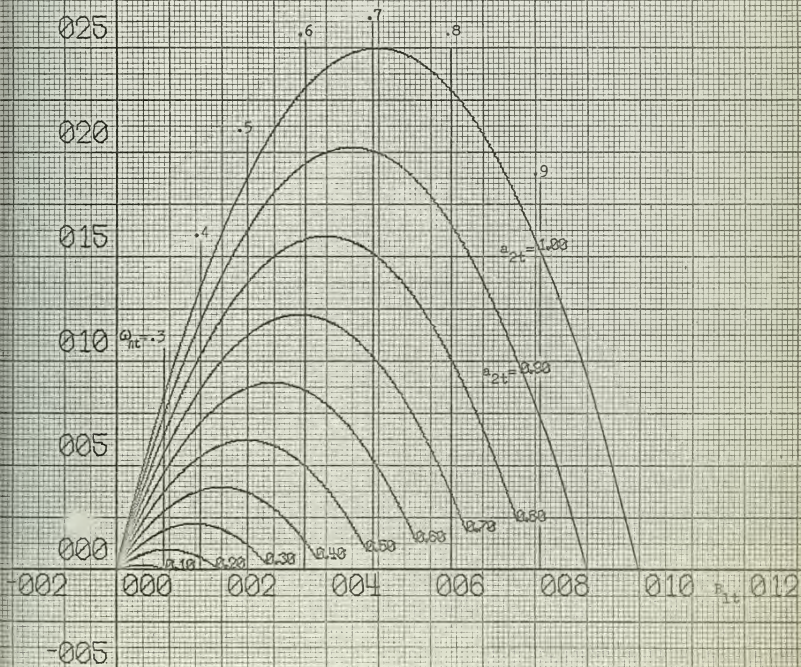
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000

Fig. 3-23  $B_{0t}$  vs  $B_{1t}$  curves (4th order systems)  
on the  $\zeta=0$  plane.



X AXIS SCALE = 2.00E-01

Y AXIS SCALE = 5.00E-02



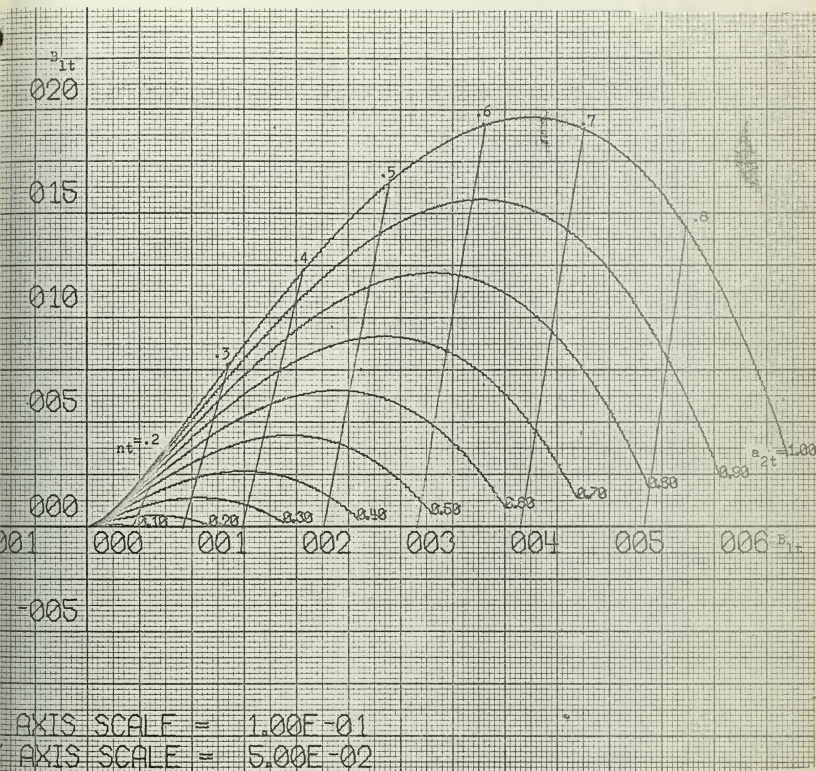


Fig. 3-24  $B_{0t}$  vs  $B_{1t}$  curves on the  $\zeta = .1$  plane.  
 (4th order systems)



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 $B_{0t}$ 

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-001

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001

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004

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-002

 $\omega_{nt} = .3$  $nt = .2$ 

0.10

0.20

0.30

0.40

0.50

0.60

0.70

0.80

0.90

1.00

Fig. 3-25  $B_{0t}$  vs  $B_{1t}$  curves (4th order systems)on the  $\zeta = .2$  plane.X AXIS SCALE =  $1.00E-01$ Y AXIS SCALE =  $2.00E-02$



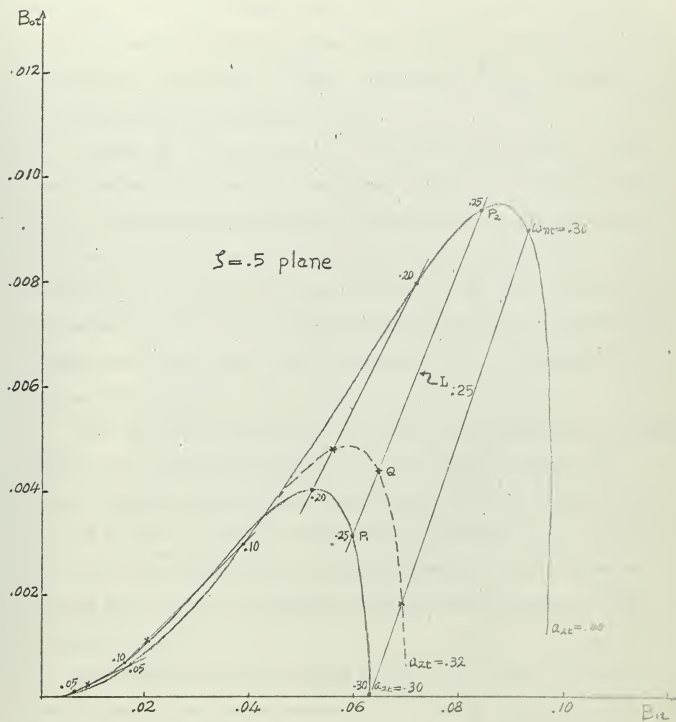


Fig. 3-26 Illustration of constructing  
a  $B_{0t}$  vs  $B_{1t}$  curve by using  
constant  $\omega_{nt}$  lines.



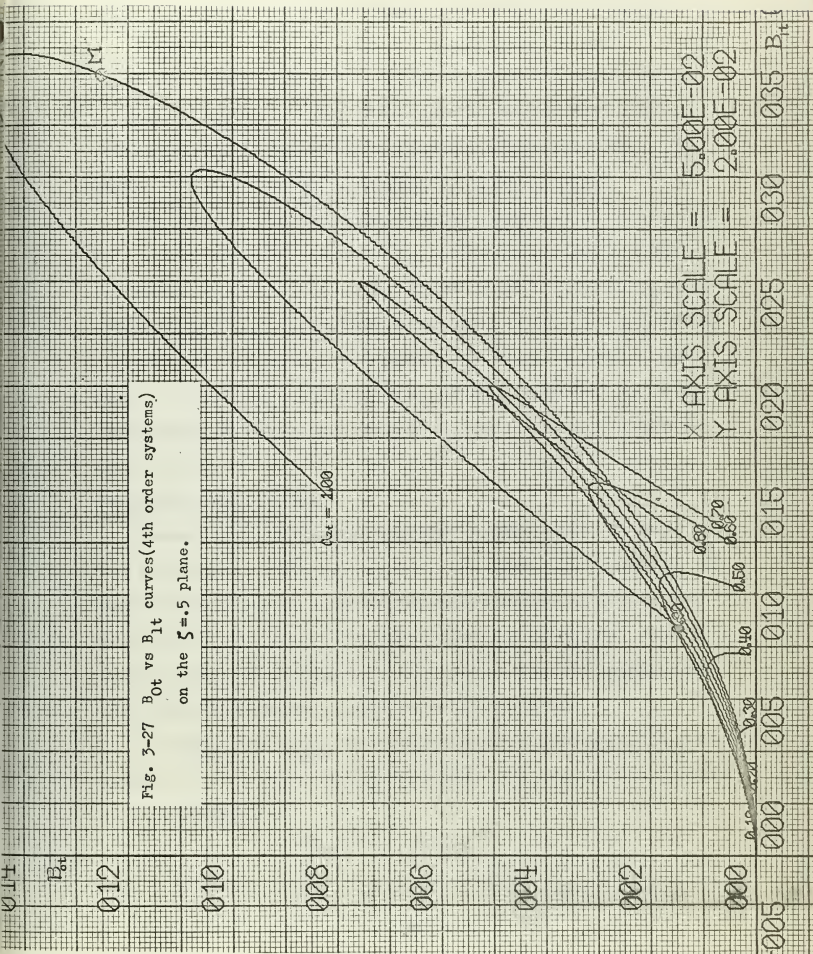
the same frequency  $\omega_{nt} = .25$  on the two curves, one for  $a_{2t} = .30$  and the other for  $a_{2t} = .40$  respectively, and the line segment joining the points  $P_1$  and  $P_2$  denoted by  $L_{.25}$  is the constant  $\omega_{nt}$  line for  $\omega_{nt} = .25$ . All members of the family  $\sqrt[.5]{.5, \{a_{2t}\}}$  will have the  $\omega_{nt} = .25$  points on or extension of the line segment  $L_{.25}$ .

Suppose the  $\sqrt[.5]{.5}$  curve for  $a_{2t} = .32$  is to be constructed. Since the increments  $\Delta B_{1t}$  and  $\Delta B_{2t}$  associated with  $\Delta a_{2t}$  are proportional to  $\Delta a_{2t}$  at every frequency, the point  $Q$  which is located at one-fifth way from the point  $P_1$  to the point  $P_2$  on the constant  $\omega_{nt}$  line  $L_{.25}$  is the point at  $\omega_{nt} = .25$  on the  $\sqrt[.5]{.5}$  curve for  $a_{2t} = .32$ . Any other points on the curve  $\sqrt[.5]{.5}$  for  $a_{2t} = .32$  are determined analogously, then connecting those points with a smooth curve, the desired curve is obtained as shown by the dotted curve.

With the aid of the constant  $\omega_{nt}$  line, one can visualize the changes of  $B_{0t}$  vs  $B_{1t}$  curves with changing  $a_{2t}$ . For example a point  $P_1$  at  $\omega_{nt} = .25$  moves along the line  $L_{.25}$  in Fig. 3-26 as  $a_{2t}$  increases from .3. If a point is located on a constant  $\omega_{nt}$  line for  $\omega_{nt} = \omega_1$  on the  $\xi = \xi_1$  plane, then the value of  $a_{2t}$  for which the  $\sqrt[.5]{\xi_1}$  curve passes through the point can be determined from the linearity property of the constant  $\omega_{nt}$  line.

Now consider an  $M$  point located at  $\omega_{nt} = \omega_{t1}$  on the  $\sqrt[.5]{\xi_1}$  curve, and one asks if the  $M$  point insures  $\xi \geq \xi_1$  for the system. For the problem to be specific, suppose the  $M$  point is chosen at  $\omega_{nt} = .5$  on  $\sqrt[.5]{.5}$  curve for  $a_{2t} = 1.0$ . The family  $\sqrt[.5]{.5, \{a_{2t}\}}$  is shown in Fig. 3-27. By applying the basic rule to determine system stability to the point  $M (.375, .125)$  with the  $\sqrt[.5]{.5}$  curve for  $a_{2t} = 1.0$ , it is obvious that the







system has a pair of complex conjugate roots with  $\zeta$  less than .5.

It is seen in Fig. 3-27 that, for  $a_{2t}$  less than or equal to .7, the entire areas enclosed by the  $\Gamma_{.5}$  curves in the first quadrant are the  $R_{.5}$  region. For  $a_{2t} = .8$  and .9 the areas enclosed by the curves are divided into two regions, one the  $R_{.5}$  region and the other the non- $R_{.5}$  region. For  $a_{2t} = 1.0$  the curve gives almost no  $R_{.5}$  region.

The curves in Fig. 3-27 agree with the curves in Fig. 3-13 to 3-22. For  $\zeta = .5$ , the critical value of  $a_{2t}$  (mentioned previously) seems to lie between  $a_{2t} = .7$  and  $a_{2t} = .8$ . If this critical value of  $a_{2t}$  is determined, then the M point chosen at any point on the  $\Gamma_{.5}$  curves for  $a_{2t}$  less than the critical  $a_{2t}$  will guarantee the system  $\zeta$  greater than or equal to .5 whenever the M point is in the first quadrant. In the following paragraphs the critical value of  $a_{2t}$  denoted by  $a_{2tc}$  is investigated for each  $\zeta$ .

#### The critical value of $a_{2t}$ .

Inspection of the figures shows that in order for a  $\Gamma_{\zeta}$  curve to form a region  $R_{\zeta}$ , the curve must encircle a point in the region  $R_{\zeta}$  in the clockwise direction. This implies as one possible situation that the points of the extrema of the coordinates  $B_{1t}$  and  $B_{0t}$  must occur in a definite order; the maximum point of  $B_{0t}$  followed by the maximum point of  $B_{1t}$  (and then minimum  $B_{0t}$  followed by minimum  $B_{1t}$ ).

Let  $\omega_{1\max}$  and  $\omega_{0\max}$  be the frequencies at which  $B_{1t}$  and  $B_{0t}$  are maximum for a  $\Gamma_{\zeta}$  curve. Then in order for a  $\Gamma_{\zeta}$  curve to form a region  $R_{\zeta}$ , the condition

$$\omega_{0\max} < \omega_{1\max}$$

must be fulfilled. (Note: even if the condition  $\omega_{0\max} < \omega_{1\max}$  is not



fulfilled, there may exist  $R_{\xi}$  and this is considered later.) This is illustrated in Fig. 3-28. In Fig. 3-28a,  $\omega_{0\max} < \omega_{1\max}$  and the region  $R_{\xi}$  is formed, whereas in Fig. 3-28b,  $\omega_{0\max} > \omega_{1\max}$  and there is no region  $R_{\xi}$ . In Fig. 3-28c the critical case  $\omega_{0\max} = \omega_{1\max}$  is shown and the region  $R_{\xi}$  is formed.

Since  $\omega_{0\max}$  and  $\omega_{1\max}$  are functions of  $a_{2t}$  and  $\xi$  in the fourth order curves, the conditions  $\omega_{0\max} \leq \omega_{1\max}$  can be expressed in terms of  $a_{2t}$  and  $\xi$  thus for a given  $a_{2t}$  the critical value of  $a_{2t}$  can be determined.

The values of  $\omega_{1\max}$  and  $\omega_{0\max}$  are found by differentiating the equations defining the  $B_{0t}$  vs  $B_{1t}$  curves Equation (3-4) which is rewritten below.

$$\begin{aligned} B_{1t} &= a_{2t}\phi_2(\xi)\omega_{nt} + \phi_3(\xi)\omega_{nt}^2 + \phi_4(\xi)\omega_{nt}^3 \\ B_{0t} &= a_{2t}\omega_{nt}^2 - \phi_2(\xi)\omega_{nt}^3 - \phi_3(\xi)\omega_{nt}^4 \end{aligned} \quad (3-4)$$

Noting that  $\phi_2(\xi) = 0$  at  $\xi = 0$ ,  $\phi_3(\xi) = 0$  at  $\xi = .5$ ,  $\phi_4(\xi) = 0$  at  $\xi = 0$ , and  $.707$ , in determining the conditions  $\omega_{0\max} \leq \omega_{1\max}$ , it is convenient to divide the interval  $0 \leq \xi \leq 1.0$  into 3 sub-intervals  $0 < \xi < .5$ ,  $.5 < \xi < .707$  and  $.707 < \xi < 1.0$ . The special cases  $\xi = 0$ ,  $.5$ ,  $.707$  and  $1.0$  are considered separately.

Differentiating  $B_{1t}$  and  $B_{0t}$  with respect to  $\omega_{nt}$  in Equation (3-4) and setting to zero

$$\begin{aligned} \frac{dB_1}{d\omega_{nt}} = 0 \quad \text{or} \quad a_{2t}\phi_2 + 2\phi_3\omega_{nt} + 3\phi_4\omega_{nt}^2 &= 0 \\ \frac{dB_0}{d\omega_{nt}} = 0 \quad \text{or} \quad 2a_{2t} - 3\phi_2\omega_{nt} - 4\phi_3\omega_{nt}^2 &= 0 \end{aligned} \quad (3-28)$$



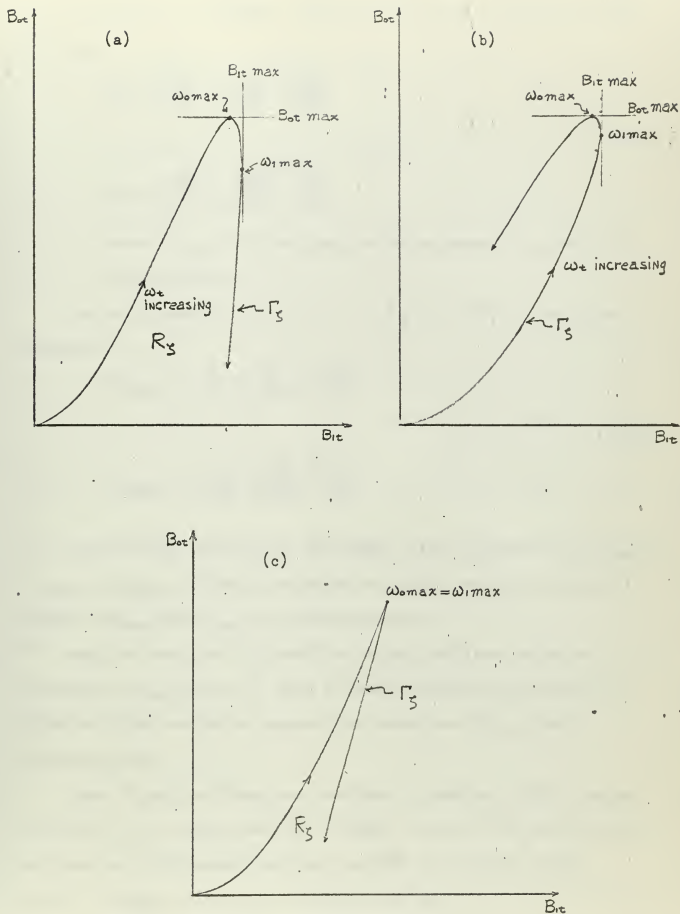


Fig. 3-28 An illustration of behavior of

$B_{0t}$  vs  $B_{1t}$  curves. (The region  $R_S$  and extreme points)



If  $\phi_3 \neq 0$  and  $\phi_4 \neq 0$ , then by solving each of Equation (3-28) for  $\omega_{nt}$

$$\omega_{1x} = -\frac{\phi_3}{3\phi_4} + \sqrt{\frac{\phi_3^2}{9\phi_4^2} - \frac{a_{2t}\phi_2}{3\phi_4}} \quad (3-29)$$

$$\omega_{0x} = -\frac{3\phi_2}{8\phi_3} + \sqrt{\frac{9\phi_2^2}{64\phi_3^2} + \frac{2a_{2t}}{4\phi_3}}$$

where  $\omega_{1x}$  is the frequency at the extremum of  $B_{1t}$  and likewise for  $\omega_{0x}$ .

For the interval of  $\xi$ ,  $0 < \xi < .5$ ,  $\phi_2 > 0$ ,  $\phi_3 > 0$  and  $\phi_4 < 0$ .

Therefore

$$\omega_{1\max} = -\frac{\phi_3}{3\phi_4} + \sqrt{\frac{\phi_3^2}{9\phi_4^2} - \frac{a_{2t}\phi_2}{3\phi_4}} \quad (3-30)$$

$$\omega_{0\max} = -\frac{3\phi_2}{8\phi_3} + \sqrt{\frac{9\phi_2^2}{64\phi_3^2} + \frac{2a_{2t}}{4\phi_3}}$$

Note that the negatives of the double signs in Equation (3-29) give  $\omega_{1\min}$  and  $\omega_{0\min}$  and those are negative numbers, and also in Equation (3-30)  $\omega_{1\max}$  and  $\omega_{0\max}$  are positive numbers.

Next one must find the conditions on  $a_{2t}$  which are equivalent to the conditions  $\omega_{0\max} \leq \omega_{1\max}$ . This is done conveniently with the aid of Equation (3-28) instead of imposing the conditions  $\omega_{0\max} \leq \omega_{1\max}$  on Equation (3-30).

Since  $\omega_{1\max}$  and  $\omega_{0\max}$  are solutions of Equation (3-28), the substitutions  $\omega_{nt} = \omega_{1\max}$  and  $\omega_{nt} = \omega_{0\max}$  into the first and the second of Equation (3-28) respectively must satisfy the Equation (3-28),

thus

$$a_{2t}\phi_2 + 2\phi_3\omega_{1\max} + 3\phi_4(\omega_{1\max})^2 = 0$$

$$2a_{2t} - 3\phi_4\omega_{0\max} - 4\phi_3(\omega_{0\max})^2 = 0$$



solving for  $a_{2t}$  and completing the squares

$$a_{2t} = -\frac{3\phi_4}{\phi_2} \left( \omega_{1max} + \frac{\phi_3}{3\phi_4} \right)^2 + \frac{\phi_3^2}{3\phi_2\phi_4} \quad (3-31)$$

$$a_{2t} = 2\phi_3 \left( \omega_{0max} + \frac{1.5\phi_2}{4\phi_3} \right)^2 - \frac{2.25\phi_2^2}{8\phi_3}$$

Equation (3-31) gives the relation between  $a_{2t}$ ,  $\omega_{1max}$  and  $\omega_{0max}$ . Noting the signs of  $\phi_2$ ,  $\phi_3$  and  $\phi_4$ , and  $-\frac{3\phi_4}{\phi_2} > 2\phi_3$  for  $0 < \xi < .5$  Equation (3-31) which are parabolas can be sketched as in Fig. 3-29.

The two equations of Equation (3-31) are plotted on the same coordinate system,  $a_{2t}$  on the ordinate and  $\omega_{1max}$ ,  $\omega_{0max}$  on the abscissa. Both parabolas pass through the origin and are concave upward since  $-3\phi_4/\phi_2 > 0$ ,  $2\phi_3 > 0$  for  $0 < \xi < .5$ . The vertices of the parabolas are below the frequency axis for both since  $\frac{\phi_3^2}{3\phi_2\phi_4} < 0$  and  $-2.25\phi_2^2/8\phi_3 < 0$  and they occur at  $\omega_{1max} = -\phi_3/3\phi_4 > 0$  and

$\omega_{0max} = -1.5\phi_2/4\phi_3 < 0$ . Thus the conditions

$$\omega_{0max} \leq \omega_{1max}$$

can be expressed in terms of  $a_{2t}$  and  $a_{2tc}$  as  $a_{2t} \leq a_{2tc}$

where  $a_{2tc}$  is the value of  $a_{2t}$  at which  $\omega_{0max} = \omega_{1max}$ . By letting

$\omega_{tc} = \omega_{1max} = \omega_{0max}$ , Equation (3-31) forms a set of two equations

with two unknown to give

$$a_{2tc} = \frac{1+\xi^2-2\xi^4}{8\xi^2(1-\xi^2)} \quad (3-32)$$

$$\omega_{tc} = \frac{1+\xi}{4\xi(1+\xi+\xi^2)}$$

By proceeding as above, it is easy to see that Equation (3-32) holds



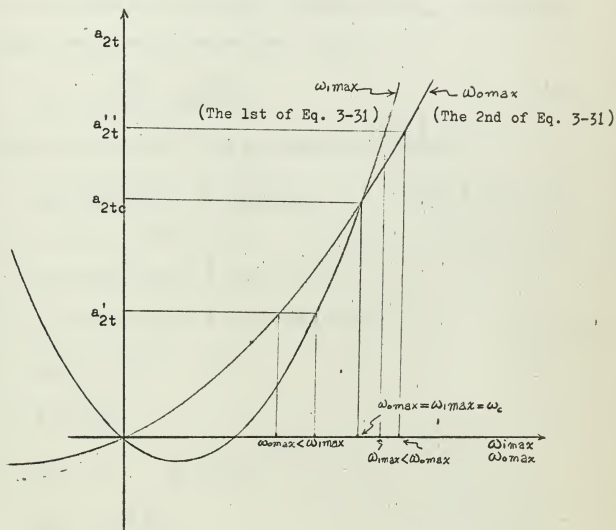


Fig. 3-29 The relation between  $a_{2tc}$ ,  $\omega_{lmax}$  and  $\omega_{0max}$ .



for the intervals  $.5 < \zeta < .707$  and  $.707 < \zeta < 1.0$ . This is because Equation (3-31) holds for any of the 3 intervals of  $\zeta$  provided  $\omega_{1\max}$  and  $\omega_{0\max}$  defined by Equation (3-30) exist. Therefore for the two intervals of  $\zeta$ ,  $.5 < \zeta < .707$  and  $.707 < \zeta < 1.0$  it is necessary to check if the value of  $a_{2tc}$  determined by Equation (3-32) gives positive real  $\omega_{1\max}$  and  $\omega_{0\max}$ .

For  $.5 < \zeta < .707$ ,  $\phi_3 < 0$ ,  $\phi_4 < 0$ , therefore from Equation (3-30),  $\omega_{1\max}$  is real positive for all  $a_{2t}$ . In order for  $\omega_{0\max}$  to be positive real  $a_{2t}$  is restricted by, noting that  $\phi_3 < 0$ ,

$$a_{2t} < -\frac{\phi_3^2}{32\phi_3} = \frac{9\zeta^2}{8(4\zeta^2-1)} \quad (3-33)$$

The expressions (3-32) and (3-33) are compared as follows:

$$a_{2tc} = \frac{1+\zeta^2-2\zeta^4}{8\zeta^2(1-\zeta^2)} \stackrel{?}{=} \frac{9\zeta^2}{8(4\zeta^2-1)} \quad \text{where } .5 < \zeta < .707$$

Since  $1-\zeta^2 > 0$ ,  $4\zeta^2-1 > 0$

$$(1+\zeta^2-2\zeta^4)(4\zeta^2-1) \stackrel{?}{\leq} 9\zeta^4(1-\zeta^2)$$

$$-8\zeta^6+6\zeta^4+3\zeta^2-1 \stackrel{?}{\leq} 9\zeta^4-9\zeta^6$$

$$\zeta^6-3\zeta^4+3\zeta^2-1 \stackrel{?}{\leq} 0$$

$$(\zeta^2-1)^3 \stackrel{?}{\leq} 0$$

Since  $\zeta^2-1 < 0$ ,  $(\zeta^2-1)^3 \leq 0$

and  $a_{2tc} < \frac{9\zeta^2}{8(4\zeta^2-1)}$

thus  $a_{2tc}$  defined by Equation (3-32) gives positive real  $\omega_{1\max}$  and  $\omega_{0\max}$  implying that Equation (3-32) is valid for the interval  $.5 < \zeta < .707$ .



The validity of Equation (3-32) for the interval  $.707 < \xi < 1.0$  can be proved likewise.

For  $\xi = 0, .5, .707$  and  $1.0$ , Equation (3-32) is also valid. If  $\xi = 0$  the  $\Gamma_0$  curves for fourth order systems form the region  $R_0$  for all values of  $a_{2t}$  implying that  $a_{2tc} \rightarrow \infty$  which agrees with Equation (3-32).

If  $\xi = .5$  then Equation (3-28) becomes

$$a_{2t} - 3\omega_{nt}^2 = 0, \quad 2a_{2t} - 3\omega_{nt} = 0$$

to give  $\omega_{max} = \sqrt{a_{2t}/3}$

$$\omega_{max} = 2a_{2t}/3$$

Let  $\omega_{1max} = \omega_{0max}$ , then  $a_{2tc} = .75$  which is equal to the value of  $a_{2tc}$  evaluated from Equation (3-32) with  $\xi = .5$

If  $\xi = .707$ , the solution of Equation (3-28) obtained likewise gives  $a_{2tc} = .5$  which is equal to the value of  $a_{2tc}$  evaluated from Equation (3-32) with  $\xi = .707$ .

If  $\xi = 1.0$  then Equation (3-32) is undefined, but  $\lim(a_{2tc}) = .375$  which agrees with the value of  $a_{2tc}$  obtained by solving Equation (3-28) with  $\xi = 1.0$

Thus a general expression of  $a_{2tc}$  as a function of  $\xi$  is obtained. Table 3.1 below which well agrees with the curves in Fig. 3-3 to 3-22 shows  $a_{2tc}$  and  $\omega_{tc}$  for some pertinent values of  $\xi$ . The  $E_{0c}$  and  $E_{1c}$  are the coordinates of the point at  $\omega_{nt} = \omega_{tc}$  on the  $\Gamma_\xi$  curve for  $a_{2t} = a_{2tc}$ .



Table 3.1

$\xi$	$a_{2tc}$	$\omega_{tc}$	$B_{0c}$	$B_{1c}$
.05	.50250E+02	.49881E+01	.62499E+03	.25000E+02
.10	.12750E+02	.24775E+01	.39050E+02	.62490E+01
.15	.58056E+01	.16347E+01	.77051E+01	.27758E+01
.20	.33750E+01	.12097E+01	.24320E+01	.15594E+01
.25	.22500E+01	.95238E+00	.99187E+00	.99584E+00
.30	.16389E+01	.77938E+00	.47532E+00	.68930E+00
.35	.12704E+01	.65486E+00	.25443E+00	.50423E+00
.40	.10313E+01	.56090E+00	.14764E+00	.38401E+00
.45	.86728E+00	.48748E+00	.91110E-01	.30159E+00
.50	.75000E+00	.42857E+00	.59038E-01	.24271E+00
.55	.66322E+00	.38032E+00	.39813E-01	.19928E+00
.60	.59722E+00	.34014E+00	.27762E-01	.16642E+00
.65	.54586E+00	.30621E+00	.19923E-01	.14102E+00
.70	.50510E+00	.27723E+00	.14661E-01	.12107E+00
.75	.47222E+00	.25225E+00	.11033E-01	.10516E+00
.80	.44531E+00	.23053E+00	.84696E-02	.92325E-01
.85	.42301E+00	.21151E+00	.66209E-02	.81865E-01
.90	.40432E+00	.19475E+00	.52617E-02	.73264E-01
.95	.38850E+00	.17990E+00	.42450E-02	.66135E-01
1.00	.37500E+00	.16667E+00	.34722E-02	.60185E-01



The necessary condition on the coefficient  $a_{2t}$  for existence of the region  $R_\xi$  on the  $B_{0t}$  vs  $B_{1t}$  plane and the design charts.

As seen in the derivation of  $a_{2tc}$  in terms of  $\xi$ , the  $\Gamma_\xi$  curve of a fourth order system for a given  $\xi$  on the  $B_{0t}$  vs  $B_{1t}$  plane has the points of extrema whose locations depend upon the value of  $a_{2t}$  with respect to  $a_{2tc}$  and the encirclements of the curve, which are closely related to the existence of the region  $R_\xi$ , depends upon the locations of the extreme points.

Another feature related to the existence of the region  $R_\xi$  is the behavior of the curve at large values of  $\omega_{nt}$ . This depends upon the value of  $\xi$  and will be described with the basic equation of the curve, Equation (3-4) which is again written below.

$$\begin{aligned} B_{1t} &= a_{2t} \phi_2(\xi) \omega_{nt} + \phi_3(\xi) \omega_{nt}^2 + \phi_4(\xi) \omega_{nt}^3 \\ B_{0t} &= a_{2t} \omega_{nt}^2 - \phi_2(\xi) \omega_{nt}^3 - \phi_3(\xi) \omega_{nt}^4 \end{aligned} \quad (3-4)$$

For  $0 < \xi \leq .5$ ,  $\phi_2(\xi) > 0$ ,  $\phi_3(\xi) \geq 0$  and  $\phi_4(\xi) < 0$  indicate that the curve tends to infinity in the third quadrant. From Equation (3-29) it is seen that there are no  $\omega_{1\min}$  and  $\omega_{0\min}$ . Therefore if  $a_{2t} \leq a_{2tc}$  then the curve vents down in the first quadrant and enters into the third quadrant after passing through the fourth quadrant, thus the entire area enclosed by the curve in the first quadrant is the region  $R_\xi$ . If  $a_{2t}$  is increased beyond  $a_{2tc}$  then the curve encircles in the counter-clockwise direction, and there exists the region  $R_\xi$  provided the curve does not enter into the second quadrant. Referring to Fig. 3-28b which is the case of  $a_{2t} > a_{2tc}$  for  $0 < \xi \leq .5$ , if  $a_{2tc}$  is not too much larger than  $a_{2t}$  then the curve enters into the third quadrant through the fourth quadrant after forming a region  $R_\xi$  in the first



quadrant. (Note that in this case the curve forms a closed loop in the first quadrant and the area enclosed by the loop is non- $R_\xi$ .) If  $a_{2t}$  is greater than the value of  $a_{2tc}$  such that the curve enters into the third quadrant by passing through the origin, then there exists no region  $R$ . This value of  $a_{2t}$  denoted by  $a_{2t0}$  can be found by setting  $B_{1t} = B_{0t} = 0$  in Equation (3-4) and solving for  $a_{2t}$ , thus one obtains  $a_{2t0}$  in terms of  $\xi$  as

$$\begin{aligned} a_{2t0} &= 1/4\xi^2 \\ \omega_{t0} &= 1/2\xi \end{aligned} \quad (3-34)$$

where  $\omega_{t0}$  is the frequency at  $B_{1t} = B_{0t} = 0$  when  $a_{2t} = a_{2t0}$

Thus for  $0 < \xi \leq .5$  the condition on  $a_{2t}$  for the existence of the region  $R_\xi$  is  $0 < a_{2t} < a_{2t0}$ . As  $a_{2t}$  approaches  $a_{2t0}$  the region  $R_\xi$  gets smaller and it is in the region near to the origin.

Noting that Equation (3-34) holds for  $0 \leq \xi \leq 1.0$ , and by comparing Equations (3-32) and (3-34), it is seen that

$$\begin{aligned} a_{2tc} &< a_{2t0} & \text{for} & & 0 < \xi < .707 \\ a_{2tc} &= a_{2t0} & \text{at} & & \xi = .707 \\ a_{2tc} &> a_{2t0} & \text{for} & & .707 < \xi < 1.0 \end{aligned} \quad (3-35)$$

The relation between  $R_\xi$ ,  $a_{2t}$  and  $a_{2t0}$  for the intervals of  $\xi$ ,  $.5 < \xi \leq .707$  and  $.707 < \xi \leq 1.0$  can be analyzed in a similar way as for the case of  $0 < \xi \leq .5$

Fig. 3-30a and 3-30b are sketches of the curves for  $.5 < \xi \leq .707$  and  $.707 < \xi \leq 1.0$  respectively with various values of  $a_{2t}$ . The curves are labeled by numbers from 1 to 5 and  $a_{2t}$  increases with the labeling number. The arrow on each curve indicates the direction of increasing  $\omega_{nt}$ . The dotted portions of the curves merely indicate the behavior of



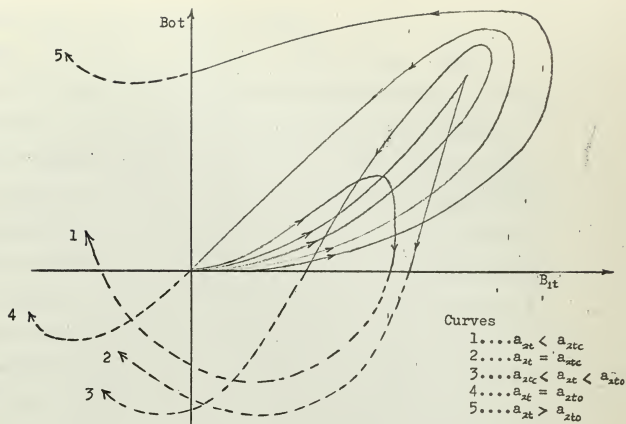


Fig. 3-30a. Illustration of the behavior of the  $B_{ot}$  vs.  $B_{it}$  curves for  $.5 < \xi \leq .707$

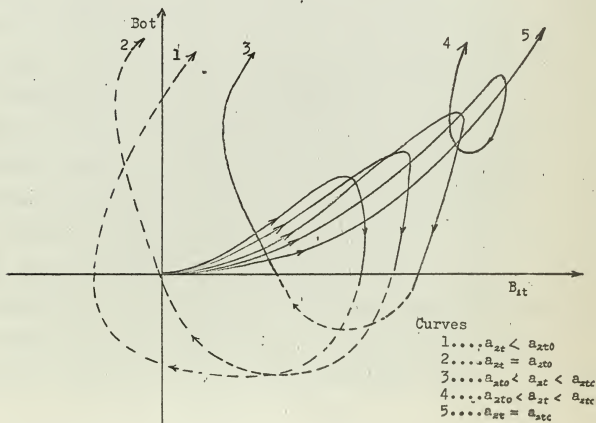


Fig. 3-30b. Illustration of the behavior of the  $B_{ot}$  vs.  $B_{it}$  curves for  $.707 < \xi \leq 1.0$



the curves after leaving the first quadrant.

For  $.5 < \zeta \leq .707$ ,  $\phi_2(\zeta) > 0$ ,  $\phi_3(\zeta) < 0$  and  $\phi_4(\zeta) < 0$

indicate the curves tend to infinity in the second quadrant. From Equation (3-29), it is seen that there is no positive real frequency of  $\omega_{1\min}$  but there may exist  $\omega_{0\min}$  for  $.5 < \zeta \leq .707$ . For  $a_{2t} \leq a_{2tc}$  which is the case of curve 1 or 2 in Fig. 3-30a the curve bends down in the first quadrant, enters into the fourth and then into the third quadrant, thus the entire area enclosed by the curve in the first quadrant is the region  $R_\zeta$ . If  $a_{2tc} < a_{2t} < a_{2to}$  then as shown by curve 3 in Fig. 3-30a, the curve encircles in the counter-clockwise direction in the first quadrant and enters into the fourth and the third then into the second, thus a region  $R_\zeta$  is formed below the earlier part of the curve. If  $a_{2t} \geq a_{2to}$  (curve 3 or 4) then the curve does not enter into the fourth quadrant and no  $R_\zeta$  is formed. Thus for  $.5 < \zeta \leq .707$ , the relation between  $R_\zeta$ ,  $a_{2tc}$  and  $a_{2to}$  is the same as for  $0 < \zeta \leq .5$ .

For  $.707 < \zeta \leq 1.0$ , the curves are sketched in Fig. 3-30b. Noting that  $a_{2to} < a_{2tc}$ , it is seen that if  $a_{2t} < a_{2to}$  (curve 1 and 2) then the entire area enclosed by the curve in the first quadrant is the region  $R_\zeta$ . If  $a_{2to} < a_{2t} < a_{2tc}$  (curve 3 and 4) then the curve forms a closed loop in the first or/and the fourth quadrant encircling in the clockwise direction to give the region  $R_\zeta$  inside the loop in the first quadrant. The curve 5 does not form the  $R_\zeta$  region, thus for  $.707 < \zeta \leq 1.0$  the condition on  $a_{2t}$  for the existence of the region  $R_\zeta$  is  $0 < a_{2t} < a_{2tc}$ .

In particular for  $\zeta = 1.0$   $a_{2tc} = .375$  from Table 3.1 and the region  $R_{1.0}$  is the region for an over-damped system. Thus the necessary condition on  $a_{2t}$  for a fourth order system to have an over-damped region



$R_{1.0}$  is  $0 < a_{2t} < .375$  which agrees with the condition derived from the statement in References 1 and 2, that is,

The necessary condition of existence of an over-damped region for the M point is to have n-2 distinct positive real roots of the polynomial  $F''(-S) = 0$  where  $F(S) = 0$  is the characteristic equation and the primes denote differentiations with respect to S.

This can be shown as follows:

For a fourth order system, the characteristic equation is

$$f(s) = s^4 + s^3 + a_{2t}s^2 + a_{1t}s + a_{0t} = 0$$

and 
$$f''(s) = 12s^2 - 6s + 2a_{2t} = 0$$

The roots of the second equation above are

$$s = .25 \pm \sqrt{(.25)^2 - a_{2t}/6}$$

In order for the roots to be two distinct positive real

$$a_{2t} < (.25)^2(6) = .375$$

The analysis of the relation between  $R_\zeta$  and  $a_{2t}$  made above can be summarized as in Table 3.2.

Fig. 3-31 to 3-44 show the families  $\overline{\zeta, \{a_{2t}\}}$  on the  $\zeta = .3, .4, .5, .6, .7, .8$  and 1.0 planes for  $a_{2t}$  up to about  $a_{2tc}$ . (For  $\zeta = .3$  and .4 plane  $a_{2t}$  is up to 1.0.) Since in a design problem, the value of  $\zeta$  is chosen deliberately at some pertinent values, and since the curves for any intermediate values of  $a_{2t}$  shown in the figures can be constructed easily by means of the constant  $\omega_{nt}$  lines (as shown previously in Fig. 3-26) and since  $a_{2t} = a_{2tc}$  is sufficient for a curve to form the region  $R_\zeta$ , the figures may be used as design charts for fourth order systems.

An inspection of the curves reveals that the arguments developed for  $a_{2tc}, a_{2to}$  and the region  $R_\zeta$  agree with the curves.



Table 3.2

( $a_{2tc}$  is given in Table 4.1 or by Equation (3-32)  
 $a_{2t0}$  is given by Equation (3-34))

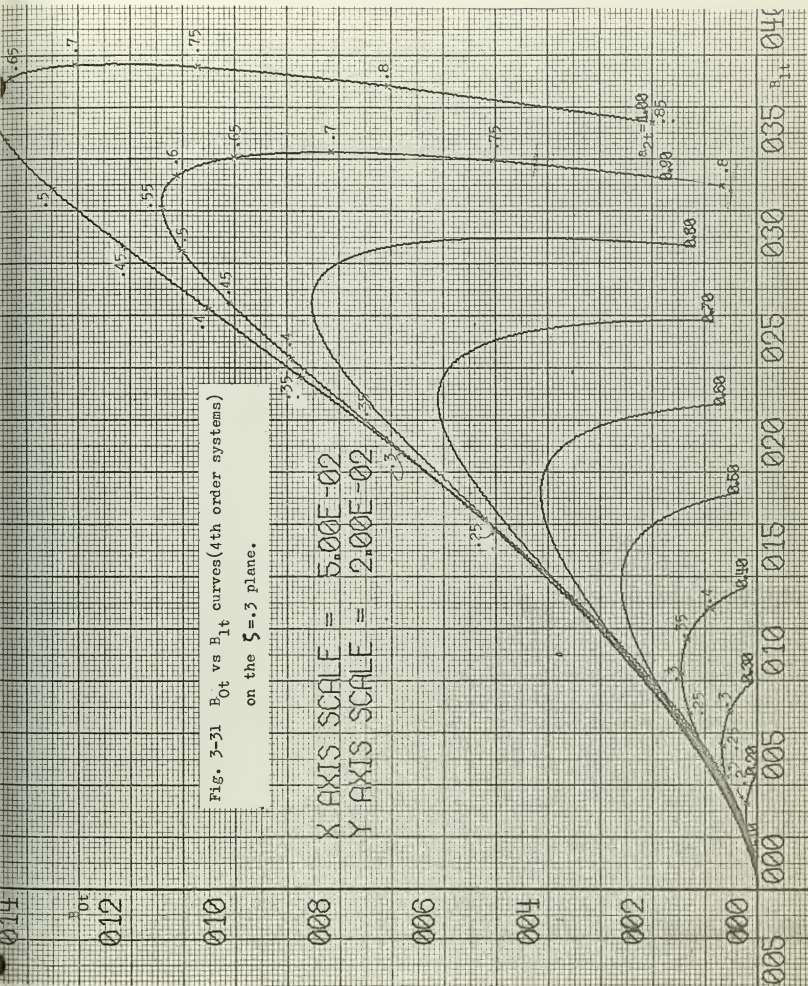
Interval of $\xi$	$a_{2t}$	$R_{\xi}$
$0 < \xi \leq .707$  $a_{2tc} < a_{2t0}$	$0 < a_{2t} \leq a_{2tc}$	Whole area in the first quadrant enclosed by the $\Gamma_{\xi}$ curve.
	$a_{2tc} < a_{2t} < a_{2t0}$	Area nearer to the origin in the first quadrant enclosed by the $\Gamma_{\xi}$ . (The area in the closed loop is non- $R_{\xi}$ )
	$a_{2t} \geq a_{2t0}$	No $R_{\xi}$
$.707 < \xi \leq 1.0$  $a_{2tc} > a_{2t0}$	$0 < a_{2t} \leq a_{2t0}$	Whole area in the first quadrant enclosed by the $\Gamma_{\xi}$ .
	$a_{2t0} < a_{2t} < a_{2tc}$	Area in the first quadrant enclosed by the closed loop encircling in the clockwise direction.
	$a_{2t} \geq a_{2tc}$	No $R_{\xi}$



Fig. 3-31  $B_{ot}$  vs  $B_{lt}$  curves (4th order systems)

on the  $\zeta = .3$  plane.

X AXIS SCALE =  $5.00E-02$   
Y AXIS SCALE =  $2.00E-02$





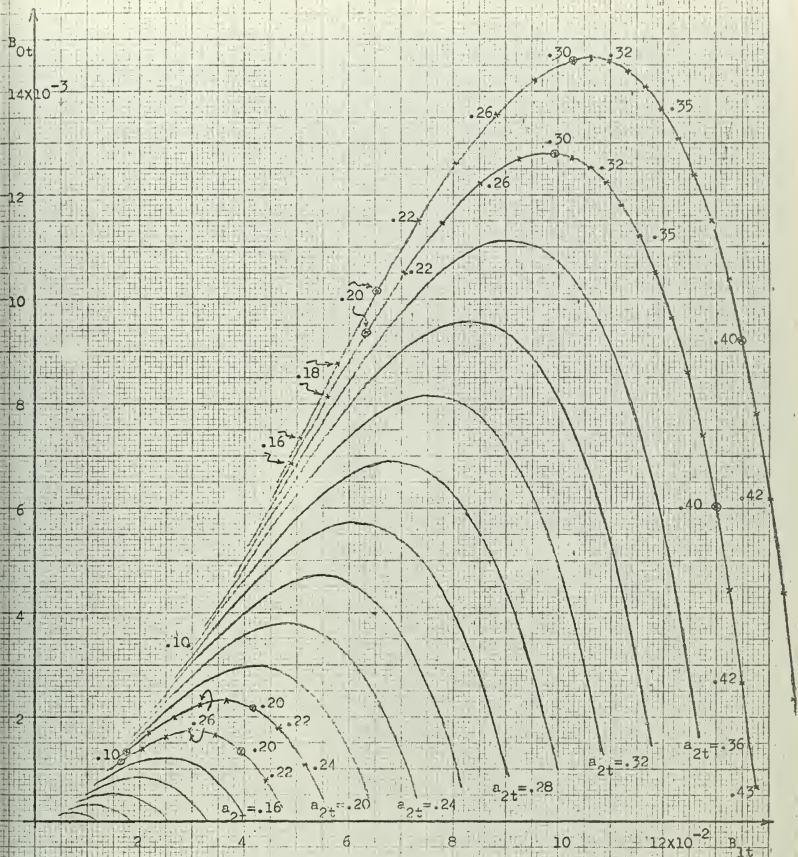
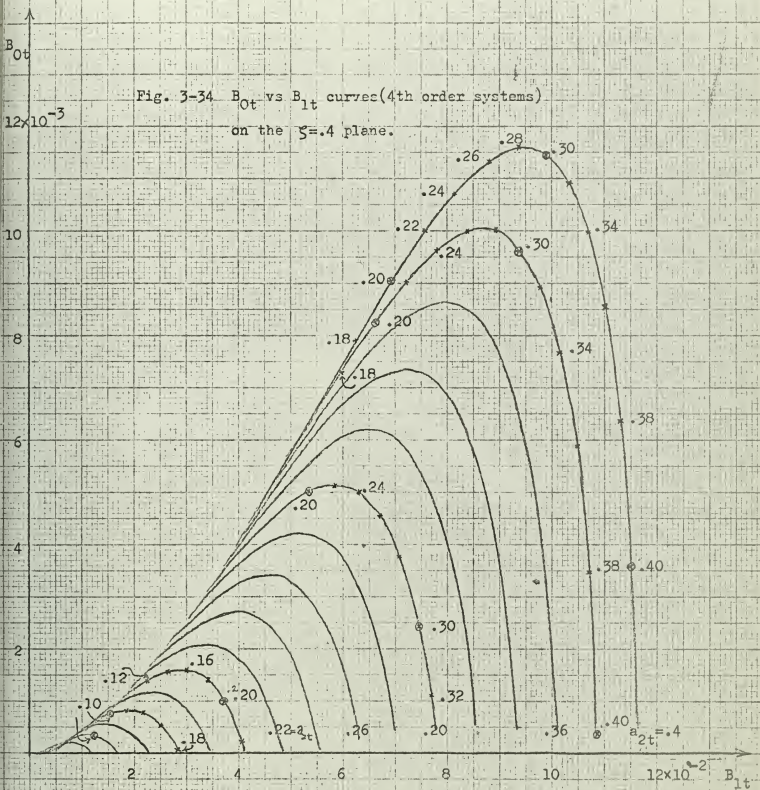


Fig. 3-32  $B_{Ot}$  vs  $B_{1t}$  curves (4th order systems)  
on the  $\zeta = .3$  plane.

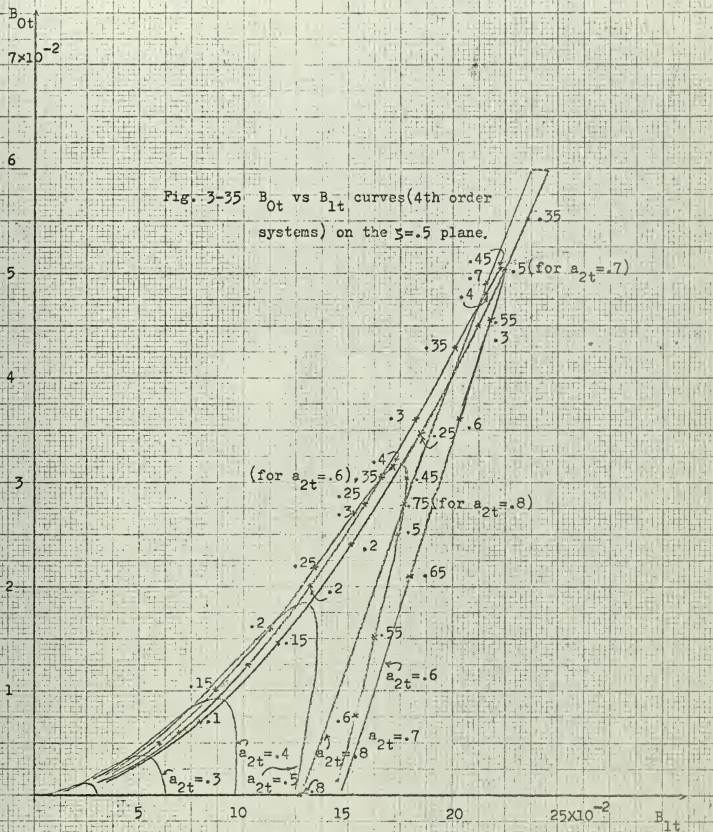




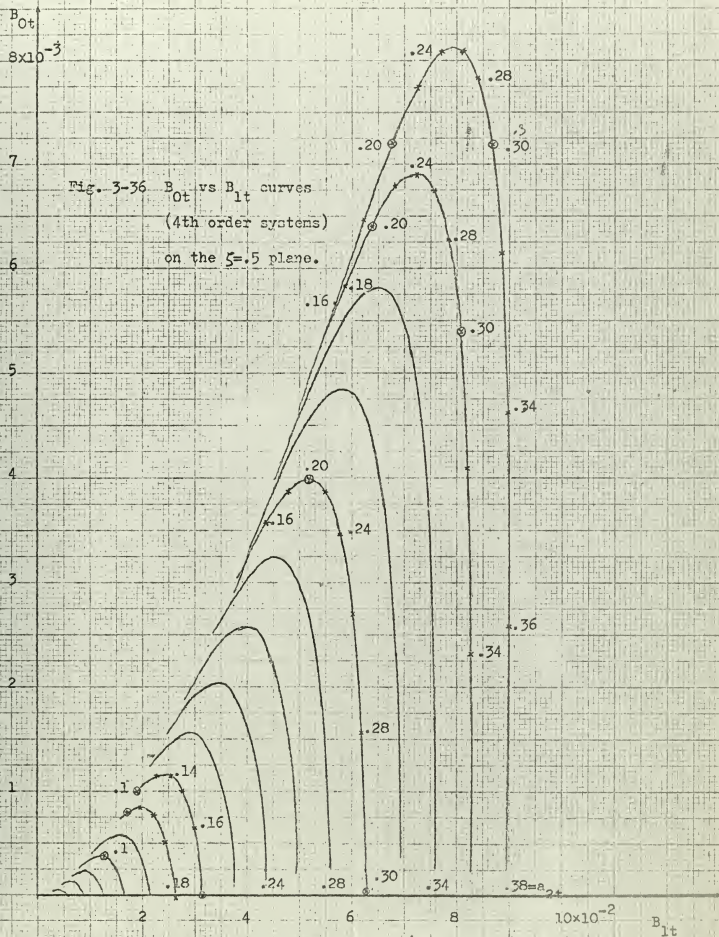














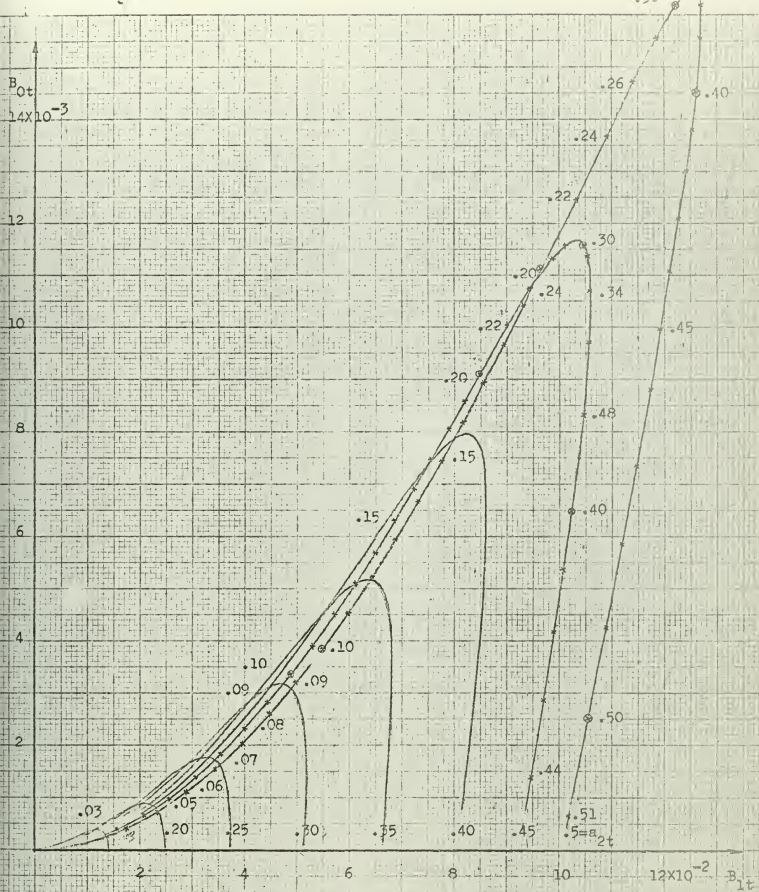
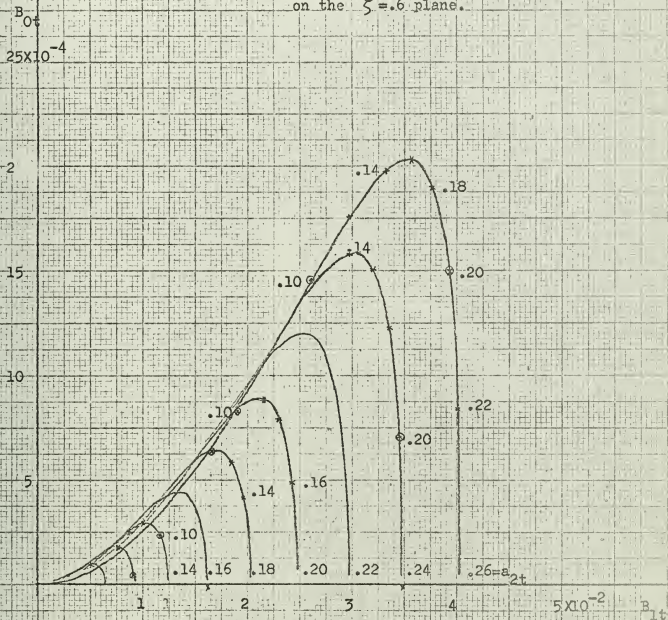


Fig. 3-37  $B_{0t}$  vs  $B_{1t}$  curves (4th order systems)  
on the  $\zeta = .6$  plane.



Fig. 3-38  $B_{0t}$  vs  $B_{1t}$  curves (4th order systems)  
on the  $\zeta = .6$  plane.





$B_{0t}$  $14 \times 10^{-3}$ 

12

10

8

6

4

2

0

2

4

6

8

10

 $12 \times 10^{-2} B_{1t}$ 

Fig. 3-39  $B_{0t}$  vs  $B_{1t}$  curves (4th order systems)  
on the  $\xi=.7$  plane.

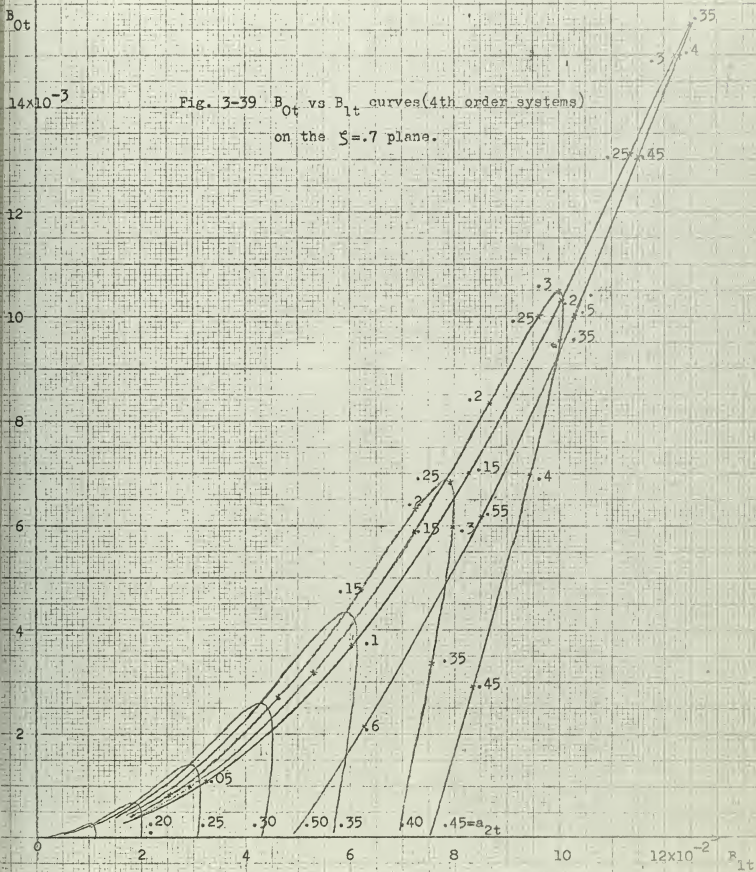
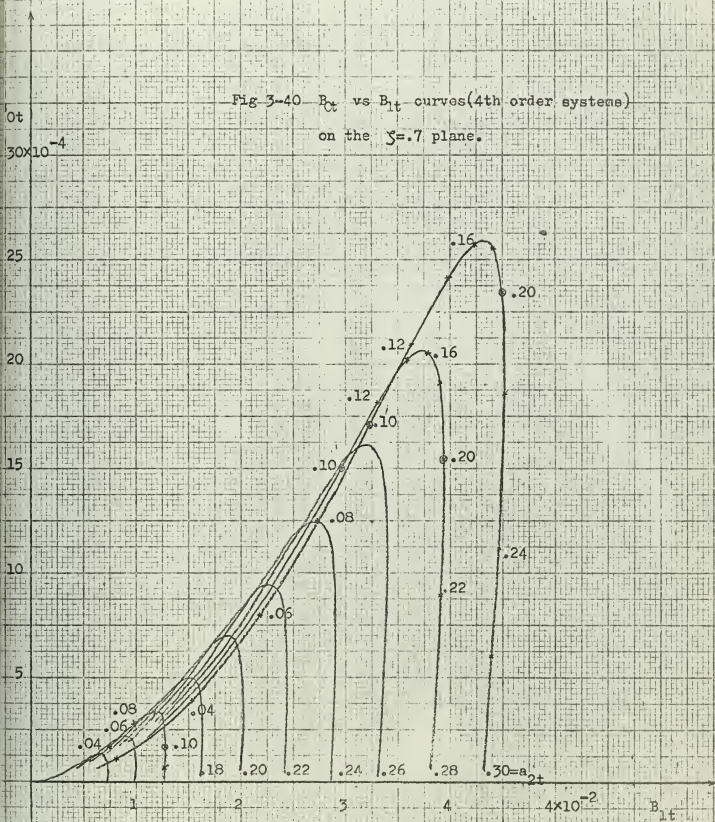




Fig 3-40  $B_{ot}$  vs  $B_{it}$  curves (4th order systems)  
on the  $\zeta=.7$  plane.





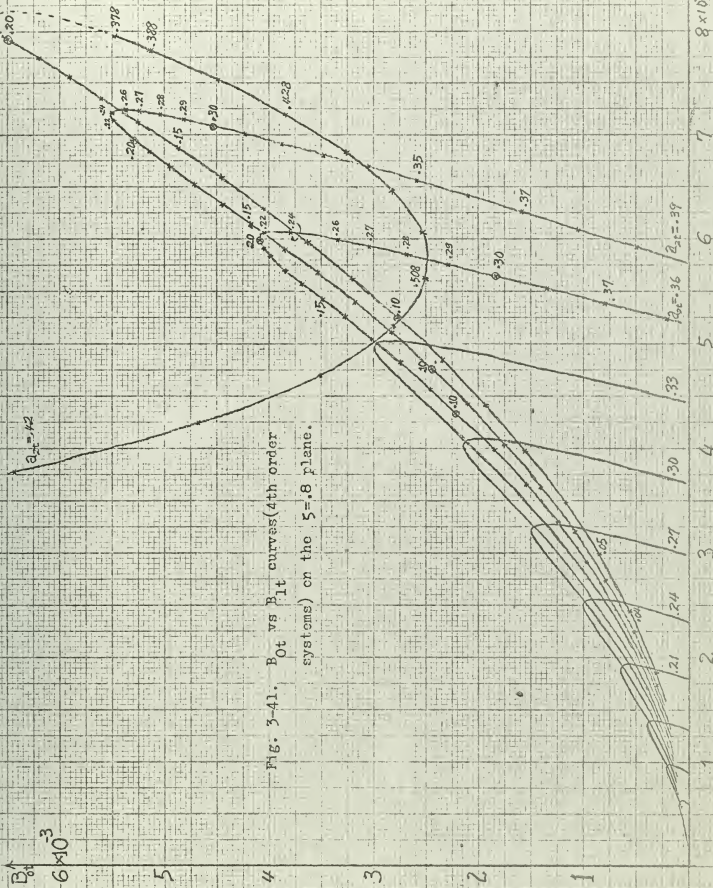
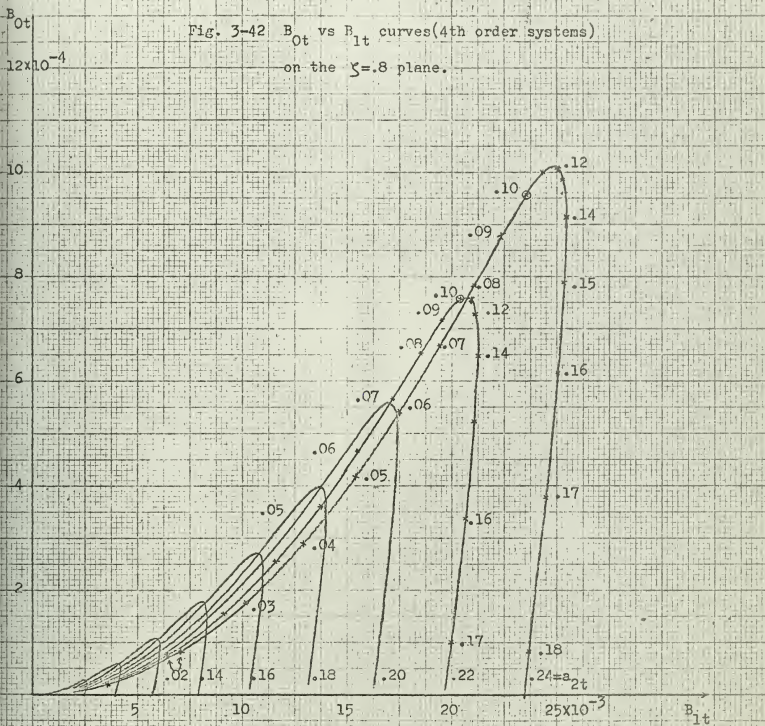
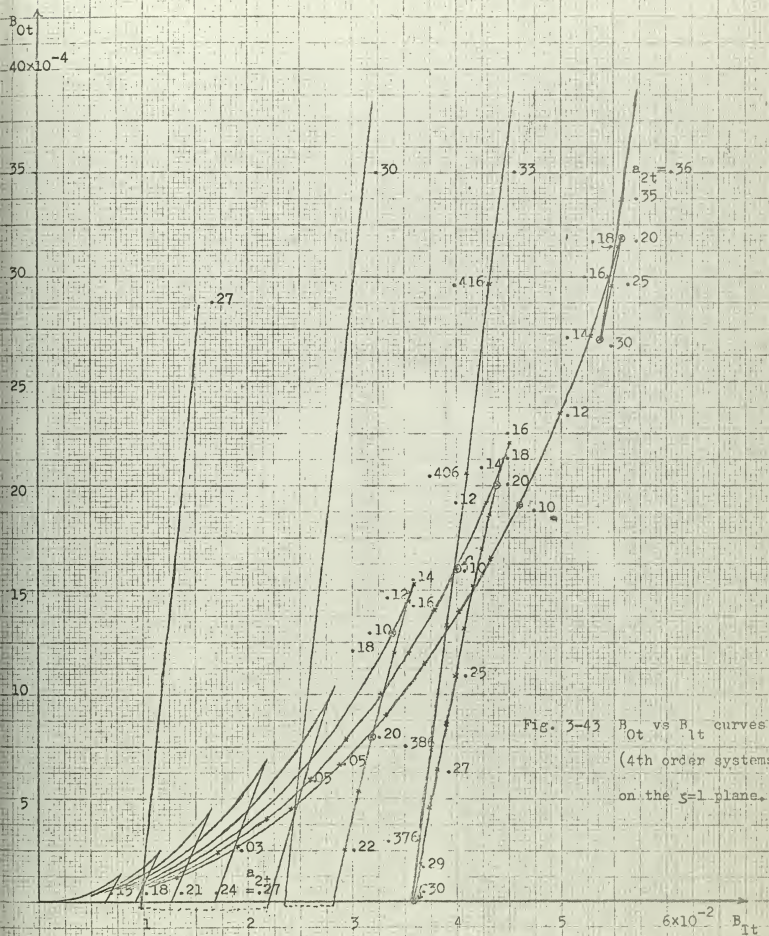


Fig. 3-41.  $B_0^2$  vs  $B_1^2$  curves (4th order systems) on the  $5=8$  plane.











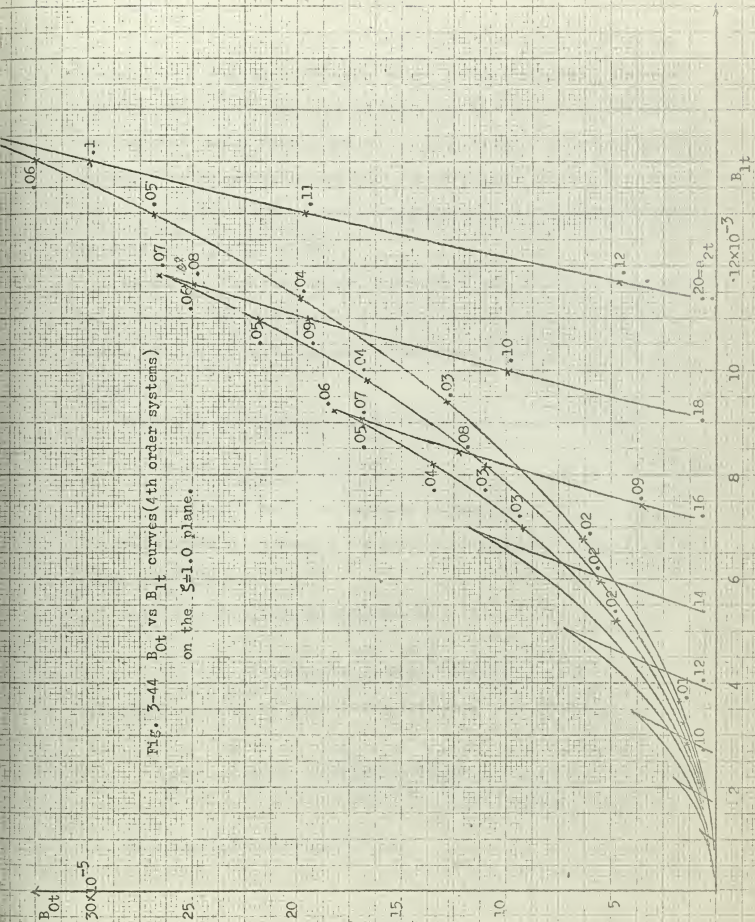


Fig. 3-44  $B_{0t}$  vs  $B_{1t}$  curves (4th order systems)  
on the  $S=1.0$  plane.



### 3-4. Application of fourth order charts.

Since the curves constructed in the previous sections are for any fourth order system and the nature of the curves analyzed is also universal to the fourth order systems, they can be used in both analysis and design of fourth order systems. It must be noted that the curves are plotted for some pertinent values of the parameters  $a_{21}$  and  $\xi$  on which the curves depend and there must be employed some interpolation techniques to apply to a specific system.

In this section, the fourth order charts are applied to some numerical examples. The procedures and methods of analysing or designing a system are the same as developed in References 1 and 2.

#### Example 3.1

Given a fourth order system with open loop transfer function

$$G(s) = \frac{K(S+Z)}{S(S^2+2S+2)(S+P)} \quad (3-36)$$

where K, Z and P are variable parameters.

The problem is to analyze the effects of varying K, Z and P on the system stability.

The closed loop transfer function is

$$\frac{C}{R}(S) = \frac{KS + KZ}{S^4 + (Z+P)S^3 + (2+2P)S^2 + (2P+K)S + KZ}$$

and transforms into, by using a transformation  $S = (2+P)s$ ,

$$\frac{C}{R}(s) = \frac{\frac{K}{(2+P)^3}s + \frac{KZ}{(2+P)^4}}{s^4 + s^3 + \frac{2+2P}{(2+P)^2}s^2 + \frac{2P+K}{(2+P)^3}s + \frac{KZ}{(2+P)^4}} \quad (3-37)$$

thus the transformed coefficients are



$$a_{0t} = \frac{KZ}{(2+P)^4}$$

$$a_{1t} = \frac{2P+K}{(2+P)^3}$$

$$a_{2t} = \frac{2+2P}{(2+P)^2}$$

$$b_{1t} = \frac{K}{(2+P)^3}$$

(3-38)

The transformed characteristic equation has unity coefficients of the terms  $s^4$  and  $s^3$  and the charts can be applied.

Suppose K and Z are fixed. By varying P,  $a_{2t}$ ,  $a_{1t}$  and  $a_{0t}$  are varied indicating that both the curves and the M point move. First consider the change of the curves. Noting that the shape of the curves (Family  $\sqrt{\xi_{12t}}$ ) depends only upon  $a_{2t}$ , the change of the curves can be analyzed by investigating the relation between  $a_{2t}$  and P.

Assuming P is any value greater than zero, from Equation (3-38)  $a_{2t}$  is a function of P. To see the dependence of  $a_{2t}$  on P,  $a_{2t}$  is differentiated with respect to P

$$\frac{da_{2t}}{dP} = -\frac{2P}{(2+P)^3} < 0 \quad \text{for } P > 0$$

implies that  $a_{2t}$  is a monotonically decreasing function of P.  $a_{2t}$  maximum is obtained at  $P = 0$  thus

$$a_{2t\max} = .5$$

In view of Table 3.1,  $a_{2tc} = .5$  is between  $\xi = .7$  and  $\xi = .75$  ( $a_{2tc} = .5$  corresponds to  $\xi = .707$ ). Noting that  $a_{2t} > a_{2tc}$  provides no region of  $R_\xi$  for  $\xi = .707$ , for very small values of P, there is no region for the M point to guarantee  $\xi \geq .707$ .

As P increases from zero,  $a_{2t}$  decreases from .5 and approaches zero when P tends to infinity. Therefore if  $\xi \geq .707$  is required then P must



be considerably larger than zero. For example if an over-damped transient response is required, then  $a_{2tc} = .375$  indicates that  $P > 1.995$ .

The locus of the M point with respect to the  $\zeta$  curves must be analyzed in order to see the system relative stability. The locus of the M points is defined by, from Equation (3-38),

$$\begin{aligned} B_{1t} &= \frac{2P+K}{(2+P)^3} \\ B_{0t} &= \frac{KZ}{(2+P)^4} \end{aligned} \quad (3-39)$$

Suppose P is fixed, then  $a_{2t}$  is fixed to give a fixed set of  $\zeta$ . The locus of the M point can be analyzed on the  $B_{0t}$  vs  $B_{1t}$  plane where the family  $\{\zeta, a_{2t}\}$  is plotted.

For the problem to be specific, suppose P is fixed to a number very close to zero. Then  $a_{2t}$  is very close to .5 and the locus of the M point for varying K and/or Z is analyzed on the chart for  $a_{2t} = .5$ . For this case, the locus of the M point is defined by

$$\begin{aligned} B_{1t} &= \frac{K}{8} \\ B_{0t} &= \frac{KZ}{16} \end{aligned} \quad (3-40)$$

which is obtained from Equation (3-39) by neglecting P.

Equation (3-40) is analyzed as follows.

If K is treated as a parameter then each value of K determines the abscissa of the M point according to the first of Equation (3-40) and the ordinate of the M point varies linearly with Z according to the second of Equation (3-40), thus the M point moves along a vertical line. Thus for a set of values of K, there results a family of vertical lines along which the M point moves.



If  $Z$  is treated as a parameter then by eliminating  $K$  from Equation (3-40), the locus of the  $M$  point becomes

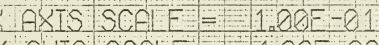
$$B_{0t} = .5Z B_{1t} \quad (3-41)$$

which is a family of lines with slopes of  $.5Z$  passing through the origin. Thus for each value of  $Z$ , the  $M$  point moves along a line defined by Equation (3-41) as  $K$  varies. In Fig. 3-45 which is a duplicate of the chart  $\sqrt{\xi}, a_{2t}$  for  $a_{2t} = .5$  the loci of the  $M$  points (the family of the lines mentioned above) are shown. If  $K = 1.0$  then the  $M$  point moves along the vertical line for  $K = 1.0$  as  $Z$  varies and if  $Z = .10$  then the  $M$  point moves along the slant line for  $Z = .10$  as  $K$  varies, thus for each value of  $K$  (or  $Z$ ), there is a limiting value of  $Z$  (or  $K$ ) for a specified system stability. For example if  $K = 1.0$  then for the system to have  $\xi \geq .5$  the value of  $Z$  must be  $Z_1 < Z < Z_2$  where  $Z_1$  is the lower bound of  $Z$  for the  $M$  point to stay inside the region  $R_{.5}$  and is found at the lower intersection point of the vertical line for  $K = 1.0$  and the  $\sqrt{.5}$  curve and analogously for  $Z_2$ . In like manner for a fixed value of  $Z$ , say  $Z = .10$ ,  $K$  must be less than  $1.5$  in order for the system to have a stability greater than  $\xi = .3$  since the intersection point of the lines for  $K = 1.0$  and  $Z = .10$  is about on the curve. Also it is seen from Fig. 3-45 that if  $K$  is greater than  $1.5$  then it would be very hard to make the system have  $\xi \geq .3$  and if  $K > 1.52$  then there is no value of  $Z$  to make the system have  $\xi \geq .3$  because the maximum abscissa of the  $\sqrt{.3}$  curve is about  $.19$  ( $K = (8)(.19) = 1.52$ )

Thus for  $P = 0$  the system stability for varying  $K$  and/or  $Z$  is analyzed. For any value of  $P$  the analysis can proceed in the same way.

Now suppose the system given by Equation (3-36) is obtained from a







third order system by cascading a filter network, that is, given a third order system open loop transfer function

$$G_o(s) = \frac{K}{s(s+2s+2)} \quad (3-42)$$

Cascade a compensator represented by

$$G_c(s) = \frac{K_c(s+z)}{s+p} \quad (3-43)$$

If  $K = 4$  then the system (3-42) is at the stability limit which can be readily seen from the third order chart and has the velocity constant  $K_v = 2$ .

Consider the problem of designing a lag compensator which is represented by Equation (3-43) so that the system stability is greater than in terms of  $\zeta$ ,  $\zeta = .5$ , with minimum allowable velocity constant  $K_v = 4$ .

The problem is to determine  $K$ ,  $Z$  and  $P$  in Equation (3-36) so that the specifications are met. By following the analysis made above and from Fig. 3-45,  $K = 1.0$ ,  $Z = .1$  and  $P \doteq 0$  will give the solution. The value of  $P$  can be determined from the  $K_v$  restriction or in a lag compensator  $P = .1Z$  (which is a reasonable pole-zero ratio as illustrated in Reference 1) gives  $P = .01$  and this  $P$  provides  $K_v = 5$  which is acceptable in this problem, thus the compensated system is

$$G(s) = \frac{1.0(s + .1)}{s(s^2 + 2s + 2)(s + .01)}$$

### Example 3.2

Consider a unity feedback servo system with its transfer function

$$G_o(s) = \frac{1000}{s(s+10)}$$

and it is desired to design a lag-lead compensator in order for the system to meet the following performance requirement:



- 1) The velocity constant  $K_v$  is to be maintained at 100.
- 2) The settling time is to be less than .5 sec.
- 3) The system is to have a  $\xi$  greater than .7.

By cascading a lag-lead compensator to the original system the compensated system becomes of fourth order and the transfer function is

$$G(S) = \frac{K(S+Z_1)(S+Z_2)}{S(S+10)(S+P_1)(S+P_2)}$$

where  $Z_1, P_1$  and  $Z_2, P_2$  are for the lag and lead sections respectively of the compensator.

There are five parameters  $K, Z_1, P_1, Z_2$  and  $P_2$  to be determined.  $K$  can be determined from the  $K_v$  specification by choosing a suitable attenuation factor of the compensator and  $P_1$  the pole of the lag section can be neglected in Mitrovic's method without danger, thus there remain three parameters  $Z_1, Z_2$  and  $P_2$  to be determined. (This kind of problem is solved in Reference 1 and in this example it is to show how the fourth order charts can be applied.)

If the compensator is chosen such that the zero to pole ratio of the lag section is to be the reciprocal of the zero to pole ratio of the lead section, then  $K$  must be set to 1000, thus the transfer function which is to be considered in the Mitrovic's method is

$$G(S) = \frac{1000(S+Z_1)(S+Z_2)}{S^2(S+10)(S+P_2)}$$

to give the characteristic equation

$$F(S) = S^4 + (10+P_2)S^3 + (1000+10P_2)S^2 + 1000(Z_1+Z_2)S + 1000Z_1Z_2 = 0$$

with the restriction on zeros and poles of the compensator as

$$\left(\frac{Z_1}{P_1}\right)\left(\frac{Z_2}{P_2}\right) = 1$$

thus there remain three parameters to be determined.



There seems to be two ways of applying the fourth order charts to solve the problem: (1) by choosing a suitable  $P_2$ , the coefficients  $a_2$ ,  $a_3$  and  $a_4$  of the characteristic equation are determined to give a fixed set of  $B_0$  vs  $B_1$  curves (for various values of  $\zeta$ ) and then one consults with the charts for the families  $\overline{\zeta, a_{2t}}$  (Fig. 3-3 to Fig. 3-22); and (2) without choosing  $P_2$  first then the coefficients  $a_2$  and  $a_3$  of the characteristic equation are variable and one consults with the charts for the families  $\overline{\zeta, a_{2t}}$  (Fig. 3-31 to Fig. 3-44). In this example the problem is solved by the first method and in the next example the second method is shown.

In order to apply the fourth order chart, the system closed loop transfer function is transformed into, by using the transformation  $S = a_3 s = (10 + P_2)s$ ,

$$\frac{C}{R}(s) = \frac{b_{2t}s^2 + b_{1t}s + a_{0t}}{s^4 + s^3 + a_{2t}s^2 + a_{1t}s + a_{0t}}$$

$$\text{where } a_{2t} = \frac{1000 + 10P_2}{(10 + P_2)^2} \quad b_{2t} = \frac{1000}{10 + P_2}$$

$$a_{1t} = \frac{1000(\xi_1 + \xi_2)}{(10 + P_2)^3}$$

$$a_{0t} = \frac{1000\xi_1\xi_2}{(10 + P_2)^4}$$
(3-44)

From the design specification (3), the M point must be inside a  $R_{.7}$  region. If one chooses  $\zeta = .8$  ( $R_{.8}$  is inside  $R_{.7}$  for a fixed  $a_{2t}$  if  $R_{.8}$  exists) then from the design specification (2) ( $\omega_n$  the frequency at the M point on the  $\overline{.8}$  curve must be greater than 10 assuming the settling time is approximately  $4/\zeta\omega_n$ , and this frequency is, in the transformed coordinates system,

$$\omega_{nt} = \omega_n / a_3 = 10 / (10 + P_2)$$



In view of Table 3.1 for the existences of the  $R_\zeta$  regions the conditions on  $a_{2t}$  are

$R_{.7}$	$a_{2t} < .505$
$R_{.8}$	$a_{2t} < .445$
$R_{.9}$	$a_{2t} < .404$
$R_{1.0}$	$a_{2t} < .375$

For the existence of  $R_{.8}$ , the equation for  $a_{2t}$  in Equation (3-44) and the above condition on  $a_{2t}$  requires  $P_2 > 49$ .

If one chooses  $a_{2t} = .3$  which is a sufficiently small number so that it provides a good flexibility to locate the M point to guarantee  $\zeta \geq .7$  then  $P_2 = 63.85$ , thus one obtains the following data

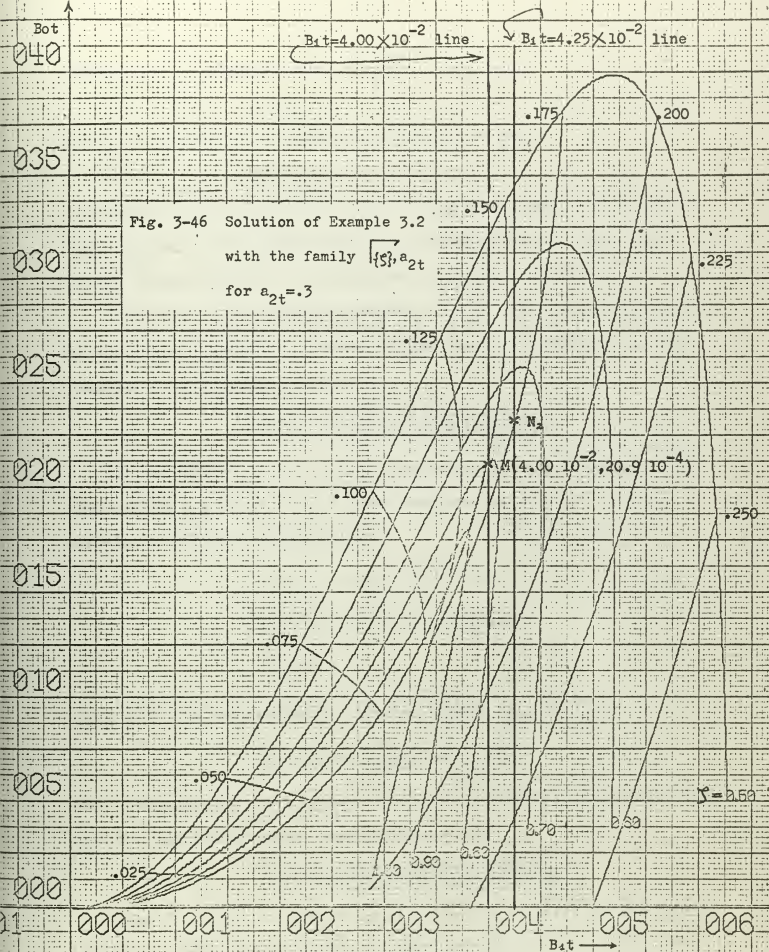
$$\begin{aligned} P_2 &= 63.85 \\ a_3 &= 73.85 \\ a_{2t} &= .3 \\ \omega_{nt} &\geq .137 \text{ on a } \Gamma_{.8} \text{ curve} \end{aligned}$$

At this point it is checked if the value of  $P_2$  determined on the basis of the region  $R_\zeta$  is sufficiently large so that the larger of the newly introduced roots of the system is much greater than the real parts of the dominant conjugate roots.

Next one consults with the family  $\Gamma_{\zeta, a_{2t}}$  for  $a_{2t} = .3$  which is in Fig. 3-8. Fig. 3-46 is a duplicate of Fig. 3-8. The curves for  $\zeta = .7, .8, .9$  and  $1.0$  with frequencies on each curve are available. It is seen on the chart that the line  $B_{1t} = 4 \times 10^{-2}$  is very close to the  $\Gamma_{.8}$  curve for  $.150 < \omega_{nt} < .200$  which is an allowable range of  $\omega_{nt}$  at the M point of this problem. Thus the abscissa of the M point is set to

$$a_{1t} = 4 \times 10^{-2}$$





X AXIS SCALE =  $1.00E-02$   
Y AXIS SCALE =  $5.00E-04$



This  $a_{1t}$  and  $P_2$  substituted into Equation (3-44) gives

$$Z_1 + Z_2 = 16$$

Let  $Z_2 = .1P_2$  then  $Z_2 = 6.385$  and  $Z_1 = 9.615$  to give

$$Z_1 Z_2 = 61.4$$

From Equation (3-44)  $a_{0t} = 20.9 \times 10^{-4}$  thus the M point is located at  $(a_{1t}, a_{0t}) = (4 \times 10^{-2}, 20.9 \times 10^{-4})$  and this point is at  $\omega_{nt} = .150$  on the  $\Gamma_{.8}$  curve which is a satisfactory M point for the problem.

Since  $Z_2 = .1 P_2$  is chosen, then  $P_1 = .1 Z_1$  must be chosen in view of the chosen compensator pole-zero ratios, thus  $P_1 = .9615$  and the compensated system transfer function is

$$G(s) = \frac{1000(s+9.615)(s+6.385)}{s(s+10)(s+.9615)(s+63.85)}$$

If one chose  $a_{1t} = 4.25 \times 10^{-2}$  then using the proportionality relation between  $a_{1t}$  and  $(Z_1 + Z_2)$  as seen from Equation (3-44),  $(Z_1 + Z_2)$  associated with  $a_{1t} = 4.25 \times 10^{-2}$  is

$$Z_1 + Z_2 = (16) \left( \frac{4.25 \times 10^{-2}}{4 \times 10^{-2}} \right) = 17$$

$$Z_1 = 17 - Z_2 = 17 - 6.385 = 10.62$$

$$Z_1 Z_2 = 68$$

again using the proportionality relation between  $a_{0t}$  and  $Z_1 Z_2$

$$a_{0t} = (20.9 \times 10^{-4}) \left( \frac{68}{61.4} \right) = 23.1 \times 10^{-4}$$

and so determined M point denoted by  $M_2$  in Fig. 3-46 is also satisfactory, then the compensated system transfer function is

$$G(s) = \frac{1000(s+10.62)(s+6.385)}{s(s+10)(s+1.062)(s+63.85)}$$

Thus if a first try is not satisfactory then a second try can be made without too much calculation and the direction whether to increase or decrease in  $a_{1t}$  in the second try can be determined by inspecting the results of the first try.



### Example 3.3

Consider the same problem of Example 3.2 with the  $\zeta$  specification as  $\zeta = 1.0$  with the settling time specification omitted. The problem can be solved on the charts of the families  $\{\zeta, a_{2t}\}$ .

By choosing some suitable pole-zero ratios of the compensator,  $Z_1$  and  $Z_2$  can be expressed in terms of  $P_2$  which is the only parameter to be determined.

By choosing a compensator such that

$$Z_2 = .1 P_2$$

$$Z_1 = .01 P_2$$

$$P_1 = .001 P_2$$

then from Equation (3-44),  $a_{2t}$ ,  $a_{1t}$  and  $a_{0t}$  which must be determined are expressed in terms of  $P_2$  only, namely

$$\begin{aligned} a_{2t} &= \frac{1000 + 10P_2}{(10 + P_2)^2} \\ a_{1t} &= \frac{110P_2}{(10 + P_2)^3} \\ a_{0t} &= \frac{P_2^2}{(10 + P_2)^4} \end{aligned} \quad (3-45)$$

The latter two of Equation (3-45) define the locus of the M points on the  $B_{0t}$  vs  $B_{1t}$  plane if a set of continuous values of  $P_2$  are supplied, and this curve represents all locations of the M points (among which a suitable M point must be selected) associated with the compensator whose pole-zero ratios are set as above.

As  $P_2$  varies, the location of the M point as well as the  $\Gamma_\zeta$  curve on the  $\zeta = \text{constant}$  plane varies. Since on a  $\zeta = \text{constant}$  plane the variation of the  $\Gamma_\zeta$  curves can be analyzed by the constant  $\omega_{nt}$  lines



and since the location of the M point moves along the curve defined by the latter two of Equation (3-45), one can find a value of  $P_2 = P_2'$  such that at  $P_2'$ , the M point is on the  $\Gamma_\xi$  curve for  $a_{2t}$  determined by  $P_2'$ .

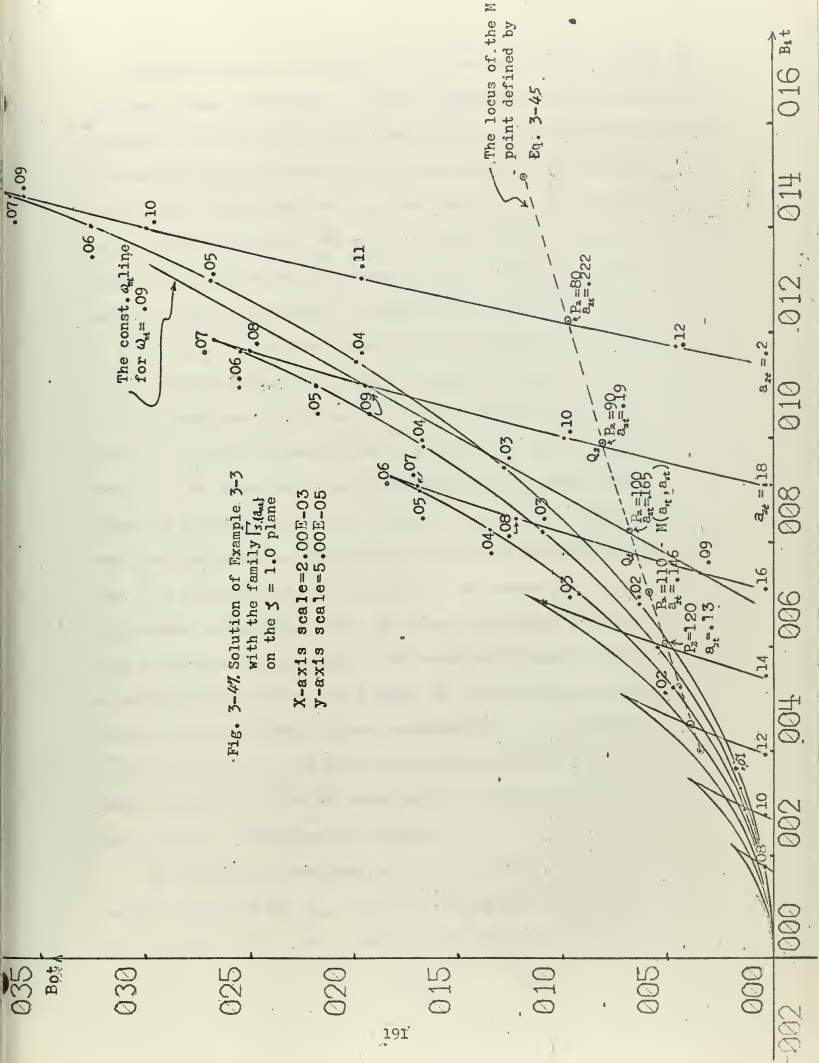
To plot the locus of the M points defined by Equation (3-45), the following table is constructed by remembering that for  $\xi = 1.0$ ,

$$a_{2t0} = .25, a_{2tc} = .375$$

$P_2$	70	80	90	100	110	120	130	140
$a_3 = 10 + P_2$	80	90	100	110	120	130	140	150
$a_{2t}$	.265	.222	.19	.165	.146	.13	.117	.10
$a_{1t}$ ( $10^{-3}$ )	15.1	12.2	9.9	8.27	7.05	6.03	5.25	4.02
$a_{0t}$ ( $10^{-5}$ )	11.9	9.75	8.1	6.85	5.85	5.05	4.4	3.44

The locus of the M point is plotted in Fig. 3-47 from the table above. Fig. 3-47 is the chart of the family  $\Gamma_{\xi, a_{2t}}$  for  $a_{2t} \leq .2$  (It is a duplicate of Fig. 3-44). The  $\Gamma_{1.0}$  curves for  $a_{2t} = .2, .18, .16, \dots$  are as shown. The frequencies on the  $\Gamma_{1.0}$  curves for  $a_{2t} = .2, .18$  and  $.16$  are labeled whereas the frequencies on the  $\Gamma_{1.0}$  curves for  $a_{2t} \leq .14$  are not shown. However the frequencies on any curve can be determined by using the constant  $\omega_{nt}$  lines; for example, if one constructs a constant  $\omega_{nt}$  line for  $\omega_{nt} = .06$  by drawing a line which passes through the two points each at  $\omega_{nt} = .06$  on the  $\Gamma_{1.0}$  curves for  $a_{2t} = .16$  and  $a_{2t} = .18$  then the line crosses the  $\Gamma_{1.0}$  curve for  $a_{2t} = .14$  at the point marked by X sign and this point is of  $\omega_{nt} = .06$  on the  $\Gamma_{1.0}$  curve for  $a_{2t} = .14$ .







On the chart with the locus of the M points plotted as shown, the relation between the family  $\sqrt{\zeta_{\{a_{2t}\}}}$  and the locus of the M points is analyzed. On each point on the locus of the M points there is associated a value of  $P_2$  which determines  $a_{2t}$ . For example if  $P_2 = 80$  then  $a_{2t} = .222$ . Noting that the  $\sqrt{1.0}$  curve for  $a_{2t} = .222$  will be shifted to the right of the curve for  $a_{2t} = .2$ , if  $P_2 = 80$  is chosen then the M point will be inside the  $\sqrt{1.0}$  curve for  $a_{2t} = .222$  implying that  $P_2 = 80$  makes the system over-damped. Consider the M point for  $P_2 = 120$ . For  $P_2 = 120$   $a_2 = .13$  and the M point will be outside of the  $\sqrt{1.0}$  curve for  $a_{2t} = .13$  since in Fig. 3-47, the  $\sqrt{1.0}$  curve for  $a_{2t} = .14$  is very close to the M point for  $P_2 = 120$  and it can be estimated that the  $\sqrt{1.0}$  curve for  $a_{2t} = .13$  will be between the  $\sqrt{1.0}$  curves for  $a_{2t} = .12$  and  $a_{2t} = .14$ , thus  $P_2 = 120$  makes the system under-damped with a fairly high  $\zeta$ . Consider the M point for  $P_2 = 100$  which gives  $a_{2t} = .165$ . Let the point  $Q_1$  and  $Q_2$  be the points at which the locus of the M points intersects with the  $\sqrt{1.0}$  curves for  $a_{2t} = .16$  and  $a_{2t} = .18$  respectively.  $a_{2t}$  of the  $\sqrt{1.0}$  curves varies from .16 to .18 between the points  $Q_1$  and  $Q_2$ . Noting that the portion of the locus of the M points between  $Q_1$  and  $Q_2$  is nearly a straight line, and that the M point for  $P_2 = 100$  is at about 1/4 way from  $Q_1$  to  $Q_2$ , by using a linear interpolation, one can predict that the  $\sqrt{1.0}$  curve for  $a_{2t} = .165$  will pass through the point M ( $a_{1t}$ ,  $a_{0t}$ ) which implies that if  $P_2 = 100$  is chosen then the M point will be on the  $\sqrt{1.0}$  curve, thus  $P_2 = 100$  gives the solution.

The frequency at the point M ( $a_{1t}$ ,  $a_{0t}$ ) can be determined with the constant  $\omega_{nt}$  line for  $\omega_{nt} = .09$  as drawn in the figure which shows that  $\omega_{nt}$  (or  $\zeta_t$ ) = .09. From the relation between transformed and



untransformed system frequencies one can obtain

$$\zeta' = a_3 \zeta_t = (10 + P_2) \zeta_t = (110)(.09) = 9.9$$

which is the value of the double real roots of the compensated system, and the compensated transfer function is

$$G(s) = \frac{1000(s+1.0)(s+10)}{s(s+10)(s+.1)(s+100)}$$

which happens to be a cancellation compensation.

It must be noted that one can construct the  $\sqrt{1.0}$  curve for  $a_{2t} = .165$  (which is for  $P_2 = 100$ ) by using the constant  $\omega_{nt}$  lines to notice that the point M ( $a_{1t}$ ,  $a_{0t}$ ) in Fig. 3-47 is slightly inside of the  $\sqrt{1.0}$  curve for  $a_{2t} = .165$ . The effect of increasing  $P_2$  on the system  $\zeta$  (against  $\zeta = 1.0$ ) can be seen from the figure, that is, by increasing  $P_2$  from 70, the system  $\zeta$  is decreased from  $\zeta > 1$  and it assumes  $\zeta \doteq 1$  at  $P_2 = 100$ . The same analysis can be made for  $P_2 < 70$  if the locus of the M points is plotted on the  $\zeta = 1.0$  plane for that range of  $P_2$ .



#### 4. Design.

In general, design or compensation of a given system is to reshape the system equation by using some compensators so that the system roots are placed at suitable locations on the S-plane with due considerations of the specified performance requirements.

In the usual design, the steady state characteristics are specified by designating the minimum allowable steady state error which can be interpreted in terms of the velocity error constant  $K_v$  (or  $K_a$ ,  $K_p$  etc. depending upon the system type number) and the transient response characteristics can be interpreted in terms of the system dominant roots locations.

In this section, assuming that the design specifications can be interpreted in terms of the steady state error constants and the locations of the dominant roots on the S-plane, an analytical as well as graphical method of designing cascaded compensators is developed by using fundamental properties of the basic Mitrovic's equations and curves in order to minimize the amount of trial and error which is usually encountered in design problems covered in the literature. Also it turns out that the labor can be considerably economized and the difficulty of choosing some suitable values of poles of the compensators (which is essential in the current Mitrovic's method) is somewhat removed.



#### 4-1. Design of single section compensator.

Consider an open loop transfer function

$$G_o(s) = \frac{K_o}{D(s)} \quad (4-1)$$

where  $K_o$  is the minimum required forward gain and  $D(s)$  is a polynomial of  $s$  with all positive coefficients,

and the design specifications require that the system dominant roots be located at

$$s = -\zeta_1 \omega_{n1} \pm j \sqrt{1 - \zeta_1^2} \omega_{n1}$$

without hurting steady state performance which is specified by the velocity constant  $K_v$  (or  $K_a$ ,  $K_p$  etc. according to the system type number).

Since the gain is adjusted to  $K_o$  the Mitrovic's working point  $M(a_1, a_0)$  associated with Equation (4-1) insures the specified velocity constant  $K_v$  but in general it does not give the roots at the specified locations. In order for the system to meet both  $K_v$  specification and the desired roots locations the point  $M(a_1, a_0)$  which guarantees the specified velocity constant must be at  $\omega_n = \omega_{n1}$  on the  $\zeta_1$  curve, and moreover the point at  $\omega_n = \omega_{n1}$  on the  $\zeta_1$  curve must be such a point that it guarantees the system  $\zeta$  not less than  $\zeta_1$ .

Let the  $B_0$  vs  $B_1$  plane be the one on which the system of interest is analyzed, and let the Mitrovic's variables for the original system in Equation (4-1) be  $B_1^o$  and  $B_0^o$  with the superscript "o" to denote the original system. Then the Mitrovic's curve for  $\zeta = \zeta_1$  of the original system is

$$\begin{aligned} E_1^o &= E_1^o(\zeta_1, \omega_n) \\ E_0^o &= E_0^o(\zeta_1, \omega_n) \end{aligned} \quad (4-2)$$



by choosing the coordinate axes such that  $B_1^0$  axis  $\equiv B_1$  axis and  $B_0^0$  axis  $\equiv B_0$  axis, the curve in Equation (4-2) can be constructed on the  $B_0$  vs  $B_1$  plane and this is denoted by  $\Gamma_{\xi_1}^0$  in Fig. 4-1 with the super-script "o" to indicate for the original system.

Let the coefficient of the  $S^1$  term in the polynomial  $D(S)$  be  $d_1$  assuming a type one system in Equation (4-1). Then the pair  $(d_1, K_0)$  determines the location of the Mitrovic's working point on the  $B_0$  vs  $B_1$  plane and it is located at

$$M^0 = M^0(d_1, K_0)$$

as shown in Fig. 4-1. The radius vector  $\vec{OM}^0$  has a fixed slope

$$\frac{K_0}{d_1} = K_v$$

and this vector is denoted by  $L'_{K_v}$ .

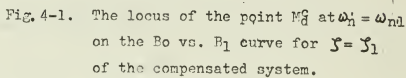
The point at  $\omega_n = \omega_{n1}$  on the  $\xi_1$  curve is located at the point denoted by  $M_d^0$  which in general does not coincide with  $M^0$  requiring a compensation. Thus the point  $M_d^0$  is the required location of the Mitrovic's working point on the  $\Gamma_{\xi_1}^0$  curve in order to meet the roots specification and the coordinates of  $M_d^0$  on the  $B_0$  vs  $B_1$  plane are, by letting  $\omega_n = \omega_{n1}$  in Equation (4-2),

$$M_d^0 = M_d^0(B_1^0(\xi_1, \omega_{n1}), B_0^0(\xi_1, \omega_{n1}))$$

The radius vector  $\vec{OM}_d^0$  has a slope  $B_0^0(\xi_1, \omega_{n1}) / B_1^0(\xi_1, \omega_{n1})$ . The  $\Gamma_{\xi_1}^0$  curve,  $M^0$  and  $M_d^0$  points configuration depends entirely upon the original system and the design specifications.

The ultimate goal is somehow to make the points  $M^0$  and  $M_d^0$  coincide since if  $M^0 \equiv M_d^0$  then both  $K_v$  and the roots specifications are met. Since there is no way to move  $M^0$  and  $M_d^0$  in Equation (4-1), a filter







network single section compensator is cascaded to the original system to give a compensated system equation

$$G(s) = \frac{K(s+Z)}{D(s)(s+P)} \quad (4-3)$$

where Z and P are zero and pole of the compensator and are variable parameters. K is the new gain which has to be adjusted to meet the steady state performance requirement.

Let  $\alpha$  be a number such that  $Z = P/\alpha$ , then from the  $K_v$  requirement K must be adjusted such that

$$K_v = \frac{K}{d\alpha} \quad \text{or} \quad K = K_v d_1 \alpha = K_o \alpha$$

Substitution of this K into Equation (4-3) gives the compensated system equation as

$$G(s) = \frac{K_o \alpha (s + P/\alpha)}{D(s)(s + P)} \quad (4-4)$$

where Z is expressed in terms of P and  $\alpha$ .

The compensated system in Equation (4-4) has its own Mitrovic's curves and Mitrovic's working point. By cascading the filter network, the significant points and curves for the original system on the  $B_0$  vs  $B_1$  plane are changed into those of the compensated system. Those changes can be analyzed mathematically as well as graphically.

Let Mitrovic's curves and the points (corresponding to  $M^o$  and  $M_d^o$  of the original system) be superscripted by "c" for the compensated system. The characteristic equation of the compensated system is

$$D(s)(s+P) + K_o \alpha (s + P/\alpha) = 0$$

$$\text{or} \quad sD(s) + PD(s) + K_o \alpha s + K_o P = 0 \quad (4-5)$$



It is seen from Equation (4-5) that any  $B_0$  vs  $B_1$  curve for the compensated system is composed of two component curves, one from the polynomial  $SD(S)$  of which all of the coefficients are fixed, thus providing a fixed curve, and the other from the polynomial  $PD(S)$  which is the original system polynomial, from which the original system curve is obtained, multiplied by  $P$ . Let  $B_1'$  and  $B_0'$  be the Mitrovic's variables for the polynomial  $SD(S)$  and let the coordinate axes be such that  $B_1'$  axis  $\equiv B_1^C$  axis  $\equiv B_1$  axis and  $B_0'$  axis  $\equiv B_0^C$  axis  $\equiv B_0$  axis, then the equation defining  $B_0$  vs  $B_1$  curve for  $\xi = \xi_1$  of the compensated system are

$$\begin{aligned} B_1^C &= B_1'(\xi_1, \omega_n) + P B_1^0(\xi_1, \omega_n) \\ B_0^C &= B_0'(\xi_1, \omega_n) + P B_0^0(\xi_1, \omega_n) \end{aligned} \quad (4-6)$$

The interpretation of Equation (4-6) on the  $B_0$  vs  $B_1$  plane is as follows. First of all the  $B_0$  vs  $B_1$  curve for  $\xi = \xi_1$  of the compensated system, namely the  $B_0^C$  vs  $B_1^C$  curve for  $\xi = \xi_1$  on the  $B_0$  vs  $B_1$  plane, depends only upon  $P$ . Assume the  $B_0'$  vs  $B_1'$  curve for  $\xi_1$  denoted by  $\Gamma_{\xi_1}'$  (which is obtained from the polynomial  $SD(S)$ ) be constructed as shown in Fig. 4-1. On the two curves  $\Gamma_{\xi_1}'$  and  $\Gamma_{\xi_1}^0$ , the points at any frequency  $\omega_n = \omega_n'$  can be located. If the coordinates of the point at  $\omega_n = \omega_n'$  on the  $\Gamma_{\xi_1}^0$  curve are multiplied by  $P$ , and added to the coordinates of the point at  $\omega_n = \omega_n'$  on the  $\Gamma_{\xi_1}'$  curve componentwise, then there locates the point at  $\omega_n = \omega_n'$  on the  $\Gamma_{\xi_1}^C$  curve. By running  $\omega_n'$  from 0 to  $\omega_n''$ , the  $\Gamma_{\xi_1}^C$  curve for the frequency range from 0 to  $\omega_n''$  can be constructed which is not shown in Fig. 4-1 since it is not necessary to construct the  $\Gamma_{\xi_1}^C$  curve at this point; instead one focuses attention to the particular point  $M_d^C$  which is at  $\omega_n = \omega_{n1}$  on the  $\Gamma_{\xi_1}^C$



curve.  $M_d^c$  is a moving point as  $P$  varies since  $M_d^c$  is at  $\omega_n = \omega_{n1}$  on the  $\Gamma_{\xi_1}^c$  curve and the  $\Gamma_{\xi_1}^c$  curve moves as  $P$  varies whereas the point  $M_d^o$  is a fixed point at  $\omega_n = \omega_{n1}$  on the  $\Gamma_{\xi_1}^o$  curve. Thus the point  $M_d^c$  is not indicated at a particular location on the  $B_0$  vs  $B_1$  plane in Fig. 4-1 but the movement of  $M_d^c$  affected by  $P$  is investigated.

By fixing the frequency at  $\omega_n = \omega_{n1}$  in Equation (4-6) the coordinate of  $M_d^c$  on the  $B_0$  vs  $B_1$  plane is determined by

$$\begin{aligned} B_1 &= B_1'(\xi_1, \omega_{n1}) + P B_1^o(\xi_1, \omega_{n1}) \\ B_0 &= B_0'(\xi_1, \omega_{n1}) + P B_0^o(\xi_1, \omega_{n1}) \end{aligned} \quad (4-7)$$

which is the equation defining the locus of the  $M_d^c$  points on the  $B_0$  vs  $B_1$  plane as  $P$  varies. Noting that  $B_1'(\xi_1, \omega_{n1})$ ,  $B_0'(\xi_1, \omega_{n1})$ ,  $B_1^o(\xi_1, \omega_{n1})$  and  $B_0^o(\xi_1, \omega_{n1})$  are fixed numbers, Equation (4-7) defines a line on the  $B_0$  vs  $B_1$  plane such that it passes through the point  $(B_1'(\xi_1, \omega_{n1}), B_0'(\xi_1, \omega_{n1}))$  denoted by  $Q$  which is the point at  $\omega_n = \omega_{n1}$  on the  $\Gamma_{\xi_1}'$  curve with a slope of  $B_0^o(\xi_1, \omega_{n1})/B_1^o(\xi_1, \omega_{n1})$  which is the slope of the radius vector  $\vec{OM_d^o}$ . This can be visualized by eliminating  $P$  in Equation (4-7) to obtain

$$B_0 - B_0'(\xi_1, \omega_{n1}) = \frac{B_0^o(\xi_1, \omega_{n1})}{B_1^o(\xi_1, \omega_{n1})} (B_1 - B_1'(\xi_1, \omega_{n1}))$$

thus the locus of  $M_d^c$  points defined by Equation (4-7) is a line parallel to  $\vec{OM_d^o}$  and passes through the point  $Q(B_1'(\xi_1, \omega_{n1}), B_0'(\xi_1, \omega_{n1}))$  on the  $B_0$  vs  $B_1$  plane. This is denoted by  $L$  in Fig. 4-1. At each point on the line  $L$ , there is associated a value of  $P$  according to Equation (4-7) and the direction of increasing  $P$  is in the direction of the radius vector  $\vec{OM_d^o}$ . At  $P = 0$ , the point at  $\omega_n = \omega_{n1}$  on the compensated  $B_0$



vs  $B_1$  curve for  $\omega = \omega_1 (M_d^C \text{ on } \Gamma_{\omega}^C)$  is at Q which is confirmed from Equation (4-7). As P increases from zero,  $M_d^C$  moves along the line L in the direction determined by  $\vec{OM}_d^0$ . Thus the desired location of the Mitrovic's working point (to insure the roots specification) which was originally at  $M_d^0$  jumps to the point Q upon cascading a single section filter network with the pole at the origin and moves along the line L as the pole of the filter network increases such that the distance from O to  $M_d^C$  is proportional to P. If a filter network with finite pole is cascaded, then it jumps from the point  $M_d^0$  to a point on the line L.

It is readily seen that the extension of the line L in the opposite direction of the vector  $\vec{OM}_d^0$  beyond the point Q corresponds to the negative P which means the pole of the compensator on the right half of the S-plane.

The orientation of the line L depends upon the locations of the point  $M_d^0$  and Q which both may be in any quadrant on the  $B_0$  vs  $B_1$  plane depending upon the original system and the design specifications.

So far the effects of cascading a single section filter network on the  $M_d^0$  point are analyzed. Remembering that in general the Mitrovic's working point determined by the pair  $(a_1, a_0)$  the coefficients of the  $s^1$  and  $s^0$  terms respectively of the characteristic equation is not identical to the desired M point located at  $\omega_n = \omega_{n1}$  on the  $\Gamma_{\omega}^C$ , thus it is necessary to adjust the variable parameters to bring the two points together to a suitable location. To do this the behavior of the point determined by the coefficients  $a_1$  and  $a_0$  must be investigated.

The point  $M^0$  is at  $(d_1, K_0)$  for the original system. By cascading the filter, this point is moved to some other location on the  $B_0$  vs  $B_1$



plane since the coefficients of the characteristic equation are changed. Let the point  $M^c$  be the point for the compensated system analogous to the point  $M^o$  of the original system. Then from Equation (4-5) the characteristic equation of the compensated system, the  $M^c$  point is defined by

$$M^c = M^c(d_1 P + K_o \alpha, K_o P)$$

which implies that the location of the  $M^c$  point on the  $B_0$  vs  $B_1$  plane depends upon both  $\alpha$  and  $P$ .

The locus of the  $M^c$  points on the  $B_0$  vs  $B_1$  plane for  $\alpha$  and  $P$  variables is defined by

$$\begin{aligned} B_1 &= d_1 P + K_o \alpha \\ B_0 &= K_o P \end{aligned} \tag{4-8}$$

By eliminating  $P$  in Equation (4-8) and noting that  $K_o = K_v d_1$ , one obtains

$$B_0 = K_v (B_1 - K_o \alpha) \tag{4-9}$$

which is an equation of lines with the slopes fixed at  $K_v$  and the  $B_1$ -axis intersections given by  $K_o \alpha$  which is a linear function of  $\alpha$ , thus the loci of the  $M^c$  points are a family of lines parallel to the line  $L'_{Kv}$ .

In Fig. 4-2, some members of this family are shown with the configuration of the line  $L$ , the vectors  $\vec{OM}^o$ ,  $\vec{OM}^c_d$  and  $Q$  point assumed to be the same as in Fig. 4-1. The line  $L'_{Kv}$  ( $\vec{OM}^o$  vector) itself is a member of the family defined by Equation (4-8) or Equation (4-9) corresponding to  $\alpha = 0$ . Any member of the family of the loci of the  $M^c$  points can be obtained by translating the line  $L'_{Kv}$  horizontally since the slope of any member is fixed at  $K_v$ , and for any two members one for  $\alpha = \alpha_1$  and the other for  $\alpha = \alpha_2$ , any two points one on each member read the same value of  $P$  if the two points are at the same ordinate, which is readily seen from the second of Equation (4-8).







Let the line  $L'_\alpha$  be a member of the family. At each point on the line  $L'_\alpha$  there associates a value of  $P$  linearly according to Equation (4-8). The point for  $P = 0$  is on the  $B_1$  axis. If  $L'_\alpha$  is divided into two parts by the  $B_1$  axis above and below, then the upper part is for  $P > 0$  and the lower part is for  $P < 0$ . Now let the line  $L'_\alpha$  move horizontally by varying  $\alpha$ . Then for  $\alpha > 0$ ,  $L'_\alpha$  is on the right of  $L'_{Kv}$  and for  $\alpha < 0$  it is on the left of  $L'_{Kv}$ , thus the line  $L'_\alpha$  is well defined for

$$-\infty < \alpha < \infty, \quad -\infty < P < \infty$$

From the definition of the point  $M_d^C$ , and the locus of  $M_d^C$  points  $L$ , if the Mitrovic's working point  $M$  is chosen at any point on the line  $L$  then the compensated system has roots at  $S = -\zeta_1 \omega_{n1} \pm \sqrt{1 - \zeta_1^2} \omega_{n1}$  and from the equations defining the line  $L'_\alpha$ , it is obvious that if the Mitrovic's working point  $M$  is chosen at any point on  $L'_\alpha$  then the compensated system satisfies the  $K_v$  requirement. Therefore in order for the compensated system to meet both the  $K_v$  and roots specifications, the Mitrovic's working point  $M$  must be chosen such that the point is on both lines  $L$  and  $L'_\alpha$  and this implies that the Mitrovic's working point  $M$  must be at the intersection point of the two lines. If one is free from the parameter  $P$  appearing at each point on both line  $L'_\alpha$  and  $L$ , and also free from any restriction on  $\alpha$  the intersection point will be located anywhere on the line  $L$  if the two lines are not parallel, since  $\alpha$  can be varied to any number, and each intersection point determines  $\alpha$  resulting infinitely many numbers of  $\alpha$ 's. However the lines  $L'_\alpha$  and  $L$  both bear  $P$  at each point on the lines according to the linear functions of  $P$  defined in Equations (4-7) and (4-8). The proper point must be chosen among the infinitely many intersection points (as mentioned above) so



that at the intersection point the values of P on  $L'_\alpha$  and L must be the same values. This is mathematically speaking the two equations, Equations (4-7) and (4-8), must be solved simultaneously. From the property of the linearity of Equations (4-7) and (4-8), there always exists one and only one solution point. Even in the case that the lines  $L'_\alpha$  and L are parallel to each other, there exists a solution. Let the solution point be  $M_s$ , then the point  $M_s$  is determined by solving Equations (4-7) and (4-8) simultaneously, namely

$$\begin{aligned} P_s &= \frac{B'_0(\xi_1, \omega_{n1})}{K_0 - B'_0(\xi_1, \omega_{n1})} \\ \alpha_s &= \frac{1}{K_0} \left[ B'_1(\xi_1, \omega_{n1}) + P_s (B'_1(\xi_1, \omega_{n1}) - d_1) \right] \end{aligned} \quad (4-10)$$

where  $P_s$  and  $\alpha_s$  are the value of P and  $\alpha$  at the solution point  $M_s$

By substituting  $P_s$  and  $\alpha_s$  back into any of Equation (4-7) or (4-8) the location of the point  $M_s$  on the  $B_0$  vs  $B_1$  plane is

$$\begin{aligned} M_s &= M_s \left( B'_1(\xi_1, \omega_{n1}) + \frac{B'_0 B'_1(\xi_1, \omega_{n1})}{K_0 - B'_0(\xi_1, \omega_{n1})}, \frac{K_0 B'_0(\xi_1, \omega_{n1})}{K_0 - B'_0(\xi_1, \omega_{n1})} \right) \\ &= M_s (d_1 P_s + K_0 \alpha_s, K_0 P_s) \end{aligned} \quad (4-11)$$

$$\text{where } B'_0 B'_1(\xi_1, \omega_{n1}) = B'_0(\xi_1, \omega_{n1}) \cdot B'_1(\xi_1, \omega_{n1})$$

So far, any restrictions to be imposed on P,  $\alpha$  and the location of  $M_s$  in view of practical situations have not been considered, and  $P_s$  and  $\alpha_s$  in Equation (4-10) can be any number from negative infinity to positive infinity and the location of  $M_s$  defined by Equation (4-11) can be anywhere on the  $B_0$  vs  $B_1$  plane.



Now consider the restrictions on  $P_s$  and  $\alpha_s$ . In view of the fact that the filter network to be realized by a passive RC network, the first restrictions on  $P_s$  and  $\alpha_s$  are

$$P_s > 0$$

$$\alpha_s > 0$$

By considering a practical RC network either too small or too large  $\alpha$  must be avoided. Thus one may restrict  $\alpha$  such that

$$.1 \leq \alpha \leq 10$$

which is a reasonable restriction.

In order for the compensated system to be a stable one, the Mitrovic's working point must be in the first quadrant, thus the  $M_s$  point must be in the first quadrant. If the solutions given by Equations (4-10) and (4-11) are within the restrictions made above, then the solutions may be the proper ones. If the solutions violate any of the restrictions above, then this is an indication that the original system cannot be compensated to meet the design specifications by a practical single section filter network. Thus one must try to cascade another section of filter network or use some other scheme for compensation.

The restrictions on the solutions can be interpreted on the  $B_0$  vs  $B_1$  plane. The restriction that requires the  $M_s$  point to be in the first quadrant is fulfilled if

$$P_s > 0$$

$$\alpha_s > 0$$

because for that  $P_s$  and  $\alpha_s$ , any point on  $L'_\alpha$  is in the first quadrant. The restriction  $.1 < \alpha < 10$  indicates that the point  $M_s$  must be on the area bounded by the Lines  $L'_{.1}$  and  $L'_{10.0}$  which are the members of the



family of the loci of the  $M^C$  points for  $\alpha = .1$  and  $\alpha = 10$  respectively, and those lines are shown in Fig. 4-2. Let  $Q'$  and  $Q''$  be the points on  $L$  at which are crossed by  $L'_{.1}$  and  $L'_{10.0}$  respectively. Then noting that the  $M_s$  point must be on the line  $L$ , the allowed locations of the  $M_s$  points are between  $Q'$  and  $Q''$  on  $L$ . The line  $L'_{.1}$  (or  $L'_{10.0}$  or both) may cross  $L$  at a point below the  $B_1$  axis. If this is the case, then the allowed locations of the  $M_s$  point must be governed by the restriction  $P > 0$ .

The existence of a solution point  $M_s$  between  $Q'$  and  $Q''$  on the line  $L$  can be determined as follows. Let the point  $Q'$  read  $P = P_\ell$  and  $P = P'_\ell$  on the lines  $L$  and  $L'_{.1}$  respectively and let the point  $Q''$  read  $P = P_u$  and  $P = P'_u$  on the lines  $L$  and  $L'_{10.0}$  respectively, then there exists the point  $M$  between  $Q'$  and  $Q''$  if and only if

$$\begin{array}{l} P_\ell > P'_\ell \quad \text{and} \quad P_u < P'_u \\ \text{or} \quad P_\ell < P'_\ell \quad \text{and} \quad P_u > P'_u \end{array}$$

This is an immediate consequence of the linear properties of the equations defining  $L'_\alpha$  and  $L$ , thus one can check whether there is a point  $M_s$  in the allowable region graphically; however it is much easier to check it by solving Equations (4-7) and (4-8) simultaneously to get the solutions in Equations (4-10) and (4-11), since the equations to be solved are simple linear equations.

When the solutions turn out not to be acceptable, then by choosing some suitable  $\alpha$  and  $P$ , say  $\alpha = \alpha_1$ ,  $P = P_1$ , to get an intermediately compensated system

$$G_1(s) = \frac{K_1 (s + z_1)}{D(s)(s + p_1)} \quad (4-12)$$

where  $K_1$  is the new gain adjusted to give the specified  $K_v$



one can follow the same procedures as developed above for the system in Equation (4-12), that is the system in Equation (4-12) is treated as the system in Equation (4-1) and the polynomial  $D(S)(S+P_i)$  in Equation (4-12) plays the role of the polynomial  $D(S)$  in Equation (4-1). By cascading another section of filter network to the system in Equation (4-12), the system of interest is then

$$G(S) = \frac{K(S+Z_i)(S+Z)}{D(S)(S+P_i)(S+P)} \quad (4-13)$$

The system in Equation (4-13) is essentially in the same form as the system in Equation (4-3) except that  $K$  which must be determined enters into the coefficient of  $S^2$  term (affecting  $B_0$  vs  $B_1$  curves) of the characteristic equation for the system in Equation (4-13). The details for this case including how to choose  $Z_i$  and  $P_i$  in the first stage will be discussed later.

If the solutions are acceptable then there still remains one and the final check, "Are the roots at  $S = -\zeta_i \omega_{n1} \pm \sqrt{1-\zeta_i^2} \omega_{n1}$  dominant?" and this is done by sketching the  $\Gamma_{\zeta_i}$  curve for the compensated system (final) to see if the point  $M$  is on the region  $R_{\zeta}$ .

The examples shown below illustrate the arguments developed above for the single section cascaded compensator.



### Example 4.1

Consider the transfer function

$$G(s) = \frac{100}{s(s+5)}$$

which has a pair of complex conjugate roots at  $s = -2.5 \pm j 9.2$ . It is to be compensated, without reducing the velocity constant  $K_v = 20$ , so that the system has a pair of complex conjugate roots at

$$s = -\zeta_1 \omega_{n1} \pm j \sqrt{1 - \zeta_1^2} \omega_{n1} = -10.5 \pm j 11$$

which is equivalent to  $\zeta_1 = .7$  and  $\omega_{n1} = 15$

$D(s) = s^2 + 5s$  and  $d_1 = 5$ ,  $K_0 = 100$ . The  $M^0$  point is at  $(5, 100)$  and the line joining the origin and the  $M^0$  point has a slope equal to  $K_v = 20$  which is the slope of the family of the lines  $L'_\alpha$  (the loci of the  $M^c$  point).

The compensated system equation is, noting that  $d_1 K_v = 100$ , by using Equation (4-4)

$$G(s) = \frac{100\alpha(s + P/\alpha)}{(s^2 + 5s)(s + P)}$$

The  $B_0$  vs  $B_1$  curves for the original system are defined by

$$B_1^0 = \phi_2(\zeta) \omega_n$$

$$B_0^0 = \omega_n^2$$

to give the location of the  $M_d^0$  point at

$$M_d^0 = M_d^0(B_1^0(\zeta=.7, \omega_n=15), B_0^0(\zeta=.7, \omega_n=15)) = M_d^0(21, 225)$$

thus the  $M_d^0$  point is located in the first quadrant and the radius vector  $\vec{OM}_d^0$  gives the direction of the line L which is determined next. Note that the points  $M^0$  and  $M_d^0$  are not identical and a compensation is needed.

The characteristic equation of the compensated system is

$$(s^3 + 5s^2) + P(s^2 + 5s) + 100\alpha s + 100P = 0$$



in which  $SD(S) = S^3 + 5S^2$  and  $PD(S) = P(S^2 + 5S)$ . The Q point is determined from the  $B_0$  vs  $B_1$  curve for the polynomial  $SD(S)$  evaluated at  $\zeta = .7$  and  $\omega_n = 15$ , that is

$$B_1' = 5\phi_2(\zeta)\omega_n + \phi_2(\zeta)\omega_n^2$$

$$B_0' = 5\omega_n^2 - \phi_2(\zeta)\omega_n^3$$

to give  $B_1'(\zeta=.7, \omega_n=15) = -111$

$$B_0'(\zeta=.7, \omega_n=15) = -3600$$

therefore the Q point is located at (-111, -3600) which is in the third quadrant. The locus of the  $M_d^C$  points, the line L is, by using Equation (4-7),

$$B_1 = -111 + P(21)$$

$$B_0 = -3600 + P(225) \quad (4-14)$$

which define the line passing through the point Q (-111, -3600) with slope of  $225/21 = 10.7$ . Note the slope 10.7 is the slope of the radius vector  $\overrightarrow{OM_Q}$ .

The locus of the point  $M^C$  is, from the characteristic equation of the compensated system

$$B_1 = 5P + 100\alpha$$

$$B_0 = 100P \quad (4-15)$$

At this point Equations (4-14) and (4-15) may be solved simultaneously to get the solutions of P and  $\alpha$  without constructing any curves or lines; however for the illustrative purpose some significant lines are drawn in Fig. 4-3. Noting that the points  $M^O$ ,  $M_d^O$  and Q are needed only for determining the slopes and the direction etc. of the lines, those points are not shown in Fig. 4-3 but the necessary portions of the lines are drawn.



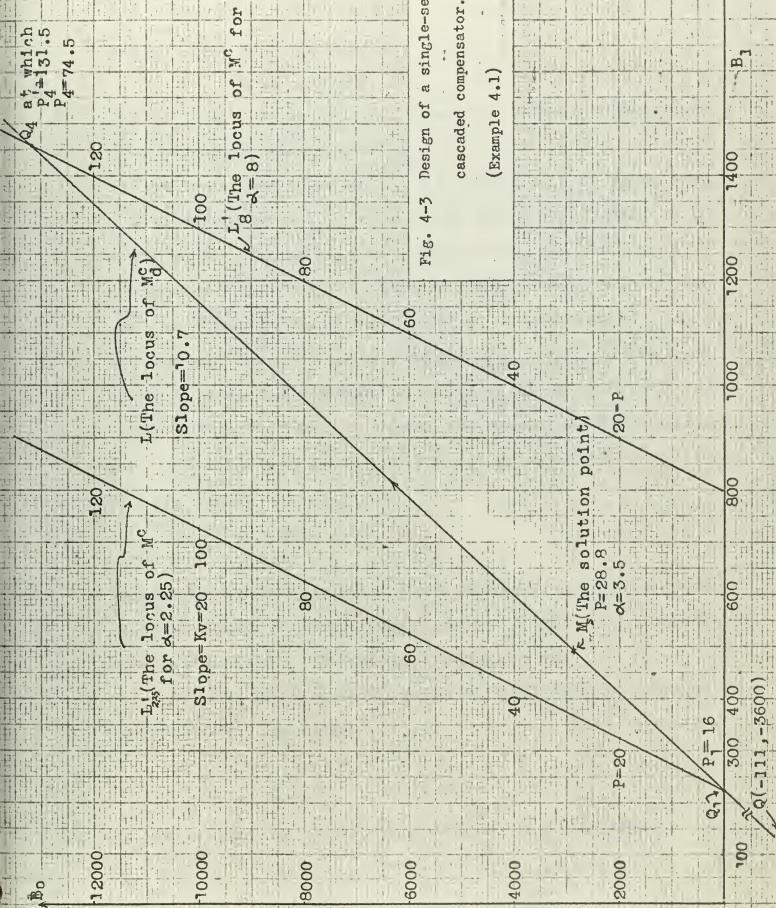


Fig. 4-3 Design of a single-section cascaded compensator.  
 (Example 4.1)



The line L is constructed as follows. From Equation (4-14) the  $B_1$  axis intersection is, by letting  $B_0 = 0$  to obtain  $P_1 = 16$  and by substituting  $P_1$  into the first of Equation (4-14),

$$B_{1 \text{ intersect.}} = 225 \quad \text{at} \quad P = 16$$

thus the point  $Q_1$  is at (225, 0). From  $Q_1$  the line L is directed into the first quadrant with slope of 10.7.

The locus of  $M^C$  point for  $\alpha = 2.25$  denoted by  $L'_{2.25}$  passes through  $Q_1$  with slope equal to  $K_v = 20$  since from the first of Equation (4-15) by letting  $B_1 = 225$  and  $P = 0$ ,  $\alpha$  is found to be 2.25. The line  $L'_8$  the locus of  $M^C$  for  $\alpha = 8$  is drawn by noting in Equation (4-15) that the  $B_1$  axis intersection of the line for  $\alpha = 8$  is 800 and  $L'_{2.25}$  and  $L'_8$  are parallel. Note that on the two lines  $L'_{2.25}$  and  $L'_8$ , for each pair of points at the same ordinates, one on  $L'_{2.25}$  and the other on  $L'_8$ , the values of  $P$  are the same. At point  $Q_4$  the lines  $L'_8$  and L intersect and read  $P'_4 = 131.5$  on  $L'_8$  and  $P_4 = 74.5$  on L. At the point  $Q_1$   $P'_1 = 0$  and  $P_1 = 16$ , thus

$$P'_1 < P_1$$

$$P'_4 > P_4$$

implies that a solution point  $M_s$  exists between  $Q_1$  and  $Q_4$  on the line L. Thus the compensation can be accomplished by a single section lead filter network with  $2.25 < \alpha < 8$  which is an acceptable one.

By solving Equations (4-14) and (4-15) simultaneously or by using Equation (4-10), one obtains

$$P_s = 28.8$$

$$\alpha_s = 3.5$$

to give the locations of the  $M_d^C$  and  $M^C$  points at  $M_s(494, 2880)$



and

$$Z_s = P_s / \alpha_s = 8.23$$

$$K = 100 \alpha_s = 350$$

then the compensated system is

$$G(s) = \frac{350(s + 8.23)}{s(s + 5)(s + 28.8)}$$

Finally the compensated system is checked on the third order chart or by constructing  $B_0$  vs  $B_1$  curves for the system to see that it is adequately compensated. Note that the characteristic equation of the compensated system factors as

$$s^3 + 33.8s^2 + 494s + 2880 = (s^2 + 21s + 225)(s + 12.8)$$

to give the roots at

$$s = -10.5 \pm j11$$

$$s = -12.8$$



### Example 4.2

Given an open loop transfer function

$$G_0(s) = \frac{6}{s(s^2 + 2.0s + 1.2)}$$

which is analyzed on the third order charts to be an unstable system with a pair of complex conjugate roots in the right half plane, design a compensator to place a pair of complex conjugate roots at  $\zeta = .3$  and  $\omega_n = .6$  with the velocity constant maintained at  $K_v = 5$ .

For the original system the significant points are

$$\begin{aligned} M^0 &= M^0(1.2, 6) \\ M_d^0 &= M_d^0(B_1^0(.3, .6), B_0^0(.3, .6)) \\ &= M_d^0(.95, .59) \end{aligned} \quad (4-16)$$

The radius vector  $\overrightarrow{OM_d^0}$  has the slope  $= .59/.95 = .63$

By cascading a filter network the compensated system is

$$G(s) = \frac{6\alpha(s + P/\alpha)}{s(s^2 + 2.5s + 1.2)(s + P)}$$

to give the characteristic equation

$$(s^4 + 2s^3 + 1.2s^2) + P(s^3 + 2.5s^2 + 1.2s) + 6\alpha s + 6P = 0$$

from which the  $B_0'$  vs  $B_1'$  curves for  $\zeta = .3$  and the Q point are obtained as

$$\begin{aligned} B_1' &= .72\omega_n + 1.28\omega_n^2 - .984\omega_n^3 \\ B_0' &= 1.2\omega_n^2 - 1.2\omega_n^3 - .64\omega_n^4 \\ B_0'(\zeta = .3, \omega_n = .6) &= .681 \\ B_0'(\zeta = .3, \omega_n = .6) &= .09 \end{aligned} \quad (4-17)$$

thus the point Q is at (.681, .09) and the locus of  $M_d^C$  point L is



defined by, using the coordinates of  $M_d^0$  and Q in Equations (4-16) and (4-17),

$$\begin{aligned} B_1 &= .681 + P(.95) \\ B_0 &= .09 + P(.59) \end{aligned} \quad (4-18)$$

From the characteristic equation of the compensated system the coefficients of the  $s^1$  and  $s^0$  terms are seen to be  $1.2P + 6\alpha$  and  $6P$  respectively, thus the locus of the  $M^c$  points is

$$\begin{aligned} B_1 &= 1.2P + 6\alpha \\ B_0 &= 6P \end{aligned} \quad (4-19)$$

By solving Equations (4-18) and (4-19) simultaneously or by using Equations (4-10) and (4-11) with the proper substitutions of the coordinates of the points  $M_d^0$  and Q from Equations (4-16) and (4-17) respectively, one obtains

$$\begin{aligned} P_s &= .0166 \\ \alpha_s &= .112 \quad Z_s = P_s / \alpha_s = .148 \quad K = 6\alpha = .673 \end{aligned}$$

and the point  $M_s$  at (.705, .0996) which are the acceptable solutions.

The compensated system is then

$$G(S) = \frac{.673(S + .148)}{S(S^2 + 2S + 1.2)(S + .0166)}$$

and can be checked on the fourth order chart for the dominance of the roots at  $\xi = .3$  and  $\omega_n = .6$

The closed loop transfer function is

$$\frac{C}{R}(S) = \frac{.673(S + .148)}{S^4 + 2.0166S^3 + 1.2332S^2 + .705S + .0996}$$

from which  $a_{2t} = \frac{a_2}{a_3} = \frac{1.223}{4} = .25$ . Noting that  $a_{2t}$  is less than  $a_{2tc}$  for  $\xi = .3$  (see Table 3.1 in Section 3-3) and the point  $M_s$  is in the first quadrant, the roots at  $\xi = .3$  and  $\omega_n = .6$  are the dominant pair.



#### 4-2. Design of double section compensators.

In a design problem, one may find that a given system cannot be compensated to meet the specified performance requirements by a single section compensator, and this is the case that the set of solutions obtained by cascading a single section filter network as in Section 4-1 turns out to be unacceptable and a multi-section compensator is needed.

Let the set of solutions obtained by cascading a single section compensator be

$$P = P_f$$

$$\alpha = \alpha_f \quad \text{and} \quad Z_f = P_f / \alpha_f$$

and suppose the solutions are not acceptable. This implies that the original system must be cascaded with a compensator with more than one section, thus the designer may try to cascade a double section compensator and proceed with exactly the same approach as in the case of a single section compensation.

The unacceptable solutions from the single section compensation are first analyzed before cascading a double section compensator. The unacceptable solutions fit into one of the following cases:

1) with  $|\alpha_f| > 1$

1a)  $P_f > 0, \quad \alpha_f > 0$

1b)  $P_f > 0, \quad \alpha_f < 0$

1c)  $P_f < 0, \quad \alpha_f > 0$

1d)  $P_f < 0, \quad \alpha_f < 0$



2) with  $|\alpha_f| < 1$

$$2a) \quad P_f > 0 \quad 0 < \alpha_f < .1$$

$$2b) \quad P_f > 0 \quad \alpha_f < 0$$

$$2c) \quad P_f < 0 \quad \alpha_f > 0$$

$$2d) \quad P_f < 0 \quad \alpha_f < 0$$

The cases b, c, d are not acceptable because of the unrealizability of the compensator with a passive network. The case 1a is for the noise problem and the restriction  $\alpha < 10$  is not a theoretical one. In case 2a which is a lag network, if  $\alpha$  is too small then the sizes of the components of the filter network may become impractical, yet some  $\alpha$  considerably smaller than .1 may be acceptable.

$$\text{Let } G_{cf}(s) = \frac{K_c(s+z_f)}{s+p_f} \quad \text{be a fictitious compensator}$$

where  $K_c$  is the attenuation factor and  $z_f = P_f / \alpha_f$ ,  $P_f$  and  $\alpha_f$  fit into one of the above cases which are the solutions obtained by cascading a single section compensator and  $G_{cf}(s)$  is not acceptable.

If  $G_{cf}(s)$  is cascaded to the original system  $G_o(s)$  in Equation (4-1), then the transfer function  $G_o(s)G_{cf}(s)$  satisfies the well-known roots locus criteria thus

$$G_o(s)G_{cf}(s) = \frac{K_o \alpha_f (s+z_f)}{D(s)(s+p_f)} = -1 \quad (4-20)$$

$$\text{at } s = -\xi_1 \omega_{n1} + \sqrt{1-\xi_1^2} \omega_{n1} = r_1$$

Noting that  $K_o > 0$ ,  $\alpha_f$  can be either positive or negative the angle criterion in Equation (4-20) for the designated root  $r_1$  is



$$\angle G_o(s) \Big|_{s=r_1} + \angle \left( \frac{\alpha_f(s+z_f)}{s+p_f} \right) \Big|_{s=r_1} = (2n-1)\pi \quad (4-21)$$

where  $n$  is an integer.

Let  $\varphi$  be the angle that must be added to the angle  $\angle G_o(s) \Big|_{s=r_1}$  by some compensator in order for the compensated system to have roots at the designated locations, then

$$\angle G_o(s) \Big|_{s=r_1} + \varphi = (2n-1)\pi \quad (4-22)$$

From Equations (4-21) and (4-22)

$$\varphi = \angle \frac{\alpha_f(s+z_f)}{s+p_f} \Big|_{s=r_1}$$

or  $\varphi = \angle \frac{s+z_f}{s+p_f} \Big|_{s=r_1} \quad \text{if } \alpha_f > 0$

$$\varphi = \angle \frac{s+z_f}{s+p_f} \Big|_{s=r_1} \quad \text{if } \alpha_f < 0 \quad (4-23)$$

Since  $0^\circ < \varphi < 180^\circ$  indicates that  $G_o(s)$  needs a lead angle in order for the compensated system to have the roots at the designated locations, and similarly since  $-180^\circ < \varphi < 0^\circ$  (or  $180^\circ < \varphi < 360^\circ$ ) indicates a lag angle, the unacceptable solutions for  $P$  and  $\alpha$  tell the designer the type of compensator needed.

With the aid of Fig. 4-4 one can obtain the following table which shows the proper type of compensator needed in view of the angle criterion for the possible combinations of the unacceptable solutions  $\alpha_f$  and  $P_f$ .



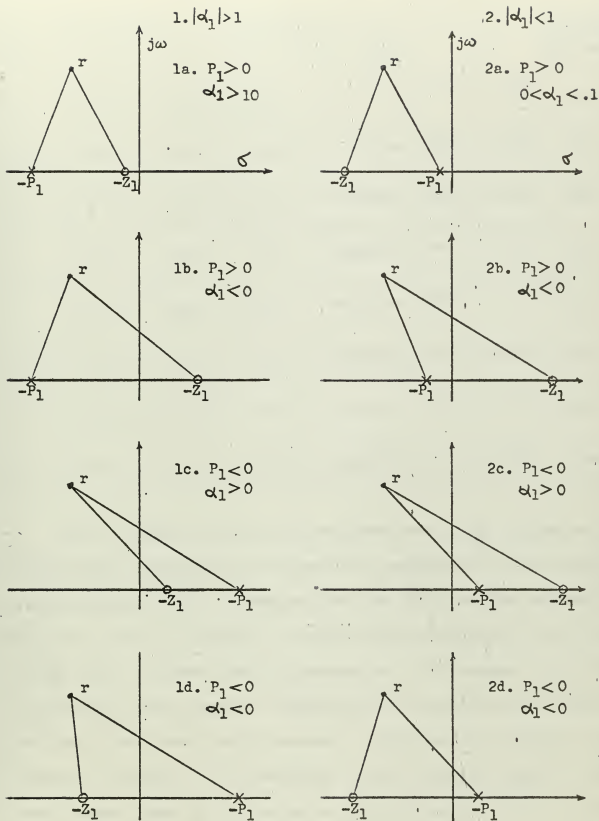


Fig. 4-4. The possible pole zero configurations of the unacceptable single section compensator,  $(S+P_1/\alpha_1)/(S+P_1)$ .



	$P_f$ and $\alpha_f$	$Z_f = P_f / \alpha_f$	$\angle \frac{s+Z_f}{s+P_f}$	$\varphi$	Indicated compensator
$ \alpha_f  > 1$	$P_f > 0, \alpha_f > 10$	$Z_f > 0$	Positive	$0^\circ < \varphi < 180^\circ$	lead
	$P_f > 0, \alpha_f < 0$	$Z_f < 0$	Positive	$180^\circ < \varphi < 360^\circ$	lag
	$P_f < 0, \alpha_f > 0$	$Z_f < 0$	Negative	$-180^\circ < \varphi < 0^\circ$	lag
	$P_f < 0, \alpha_f < 0$	$Z_f > 0$	Negative	$0^\circ < \varphi < 180^\circ$	lead
$ \alpha_f  < 1$	$P_f > 0, \alpha_f < 1$	$Z_f > 0$	Negative	$-180^\circ < \varphi < 0^\circ$	lag
	$P_f > 0, \alpha_f < 0$	$Z_f < 0$	Positive	$180^\circ < \varphi < 360^\circ$	lag
	$P_f < 0, \alpha_f > 0$	$Z_f < 0$	Positive	$0^\circ < \varphi < 180^\circ$	lead
	$P_f < 0, \alpha_f < 0$	$Z_f > 0$	Negative	$0 < \varphi < 180^\circ$	lead

Fig. 4-4 shows the pole zero configurations on the S-plane of the unacceptable compensator for all possible combinations of the unacceptable solutions  $P_f$  and  $\alpha_f$  obtained from the single section compensation. Thus the results of cascading a single section compensator to the original system gives the proper type of compensator to be cascaded. Upon realizing that the system cannot be compensated by a single section compensator, one may try as a next step to cascade a double section (identical) compensator in which it is not necessary to know the proper type of compensator needed, since if the system can be compensated by such a compensator, then the solutions will now be acceptable and the type of compensator will be indicated by the solutions.

Consider the original system in Equation (4-1) cannot be compensated



by a single section compensator and an identical double section compensator is cascaded to give the compensated transfer function

$$G(s) = \frac{K (s + P/\alpha)^2}{D(s) (s + P)^2} \quad (4-24)$$

where  $D(s)$ ,  $P$  and  $\alpha$  are defined as in Equation (4-1) with the same design specifications as in Section 4-1. From the steady state performance specification the gain  $K$  now must be adjusted to give

$$K_v = \frac{K}{d_1 \alpha^2}$$

or 
$$K = d_1 K_v \alpha^2 = K_0 \alpha^2$$

The characteristic equation of the system in Equation (4-24) with  $K$  replaced by  $K_0 \alpha^2$  can be written as

$$F(s) = D(s)(s + P)^2 + K_0 \alpha^2 (s + P/\alpha)^2 = 0$$

or 
$$F(s) = s^2 D(s) + 2PSD(s) + P^2 D(s) + K_0 \alpha^2 s^2 + 2K_0 \alpha P s + K_0 P^2 = 0 \quad (4-25)$$

An inspection of Equation (4-25) shows that the  $B_0$  vs  $B_1$  curve for  $\xi = \xi_1$  associated with the polynomial  $F(s)$  consist of the curves for the component polynomials  $s^2 D(s)$ ,  $SD(s)$ ,  $D(s)$  and  $s^2$ . The curves associated with the polynomial  $SD(s)$  and  $D(s)$  are computed already in the single section compensation as the  $B_0'$  vs  $B_1'$  and  $B_0''$  vs  $B_1''$  curves respectively. The curve associated with the polynomial  $s^2$  can be computed simply by considering  $s^2$  to be a second order polynomial of the form  $F(s) = s^2 + a_1 s + a_0$ , for which Mitrovic's equations are  $B_1 = \phi_2(\xi) \omega_n$ ,  $B_0 = \omega_n^2$ . If the curve associated with the polynomial  $s^2 D(s)$  is denoted by the  $B_0''$  vs  $B_1''$  curve then the  $B_0$  vs  $B_1$  curve for  $\xi = \xi_1$  associated with the compensated system characteristic equation, Equation (4-25) is



$$\begin{aligned}
 B_1^c &= B_1^c(\zeta_1, \omega_n) + 2P B_1^c(\zeta_1, \omega_n) + P^2 B_1^c(\zeta_1, \omega_n) + K_0 \alpha^2 \phi_2(\zeta_1) \omega_n \\
 B_0^c &= B_0^c(\zeta_1, \omega_n) + 2P B_0^c(\zeta_1, \omega_n) + P^2 B_0^c(\zeta_1, \omega_n) + K_0 \alpha^2 \omega_n^2
 \end{aligned}
 \tag{4-26}$$

which is a curve that depends upon  $P$  and  $\alpha$ . (Note that in the single section compensation, the compensated system curve depends only on  $P$ .) The point  $M_d^c$  which is at  $\omega_n = \omega_{n1}$  on the compensated system curve in Equation (4-26) moves if either  $P$  or  $\alpha$  is varied, and the equation of the movement or the locus of the  $M_d^c$  points is obtained by fixing the frequency at  $\omega_n = \omega_{n1}$  in Equation (4-26), that is, the locus of the  $M_d^c$  point on the  $B_0$  vs  $B_1$  plane is

$$\begin{aligned}
 B_1 &= B_1^c(\zeta_1, \omega_{n1}) + 2P B_1^c(\zeta_1, \omega_{n1}) + P^2 B_1^c(\zeta_1, \omega_{n1}) + K_0 \alpha^2 \phi_2(\zeta_1) \omega_{n1} \\
 B_0 &= B_0^c(\zeta_1, \omega_{n1}) + 2P B_0^c(\zeta_1, \omega_{n1}) + P^2 B_0^c(\zeta_1, \omega_{n1}) + K_0 \alpha^2 \omega_{n1}^2
 \end{aligned}
 \tag{4-27}$$

Note that in setting up Equation (4-27) only the first and last terms need new calculations.

The coefficients  $a_1$  and  $a_0$  of the terms  $S^1$  and  $S^0$  in the compensated system characteristic equation  $F(S)$  in Equation (4-25) are  $d_1 P^2 + 2K_0 \alpha P$  (The  $d_1 P^2$  is obtained from the expansion of the term  $P^2 D(S)$ ) and  $K_0 P^2$  respectively, and the locus of the  $M^c$  point for the compensated system in Equation (4-24) is defined by

$$\begin{aligned}
 B_1 &= d_1 P^2 + 2K_0 \alpha P \\
 B_0 &= K_0 P^2
 \end{aligned}
 \tag{4-28}$$

Equations (4-27) and (4-28) can be solved simultaneously for  $\alpha$  and  $P$  which are the solutions for the double section compensator. Since Equations (4-27) and (4-28) are of second order in  $P$  and  $\alpha$ , the simultaneous solution of the two equations requires the solution of a fourth



order polynomial which is in general not simple to solve; however there are some convenient methods to solve the equation such as graphical methods, approximation technique etc.

If the solutions from double section compensation are not acceptable, then a triple section compensation may be tried with the same procedure as for the double section compensation; however the results require solution of a sixth order polynomial which may be solved by a digital computer.

The following example illustrates the double section compensation.

#### Example 4.3

Given an open loop transfer function

$$G_o(s) = \frac{420}{s(s+1)(s+15)}$$

Compensate so that roots are at

$$\xi = .5$$

$$\omega_n = 15$$

without reducing the velocity constant  $K_v = 28$

The given system is unstable with the complex conjugate roots in the right half of the S-plane which can be seen from the third order chart. If a single section compensator is cascaded, then the system transfer function becomes

$$G(s) = \frac{420\alpha(s+P/\alpha)}{s(s+1)(s+15)(s+P)}$$

from which the characteristic equation is written as

$$(s^4 + 16s^3 + 15s^2) + P(s^3 + 16s^2 + 15s) + 420\alpha s + 420P = 0$$

The polynomial  $SD(s) = s^4 + 16s^3 + 15s^2$  gives the  $B_0'$  vs  $B_1'$  curve for

$$\xi = .5 \text{ as}$$



$$B_1'(s=5, \omega_n) = 15\omega_n - \omega_n^3$$

$$B_0'(s=5, \omega_n) = 15\omega_n^2 - 16\omega_n^3$$

from which  $B_1'(s=5, \omega_n=15) = -3150$

$$B_0'(s=5, \omega_n=15) = -50625 \quad (4-29)$$

are obtained.

The polynomial  $D(s) = s^3 + 16s^2 + 15s$  gives

$$B_1^0(s=5, \omega_n) = 16\omega_n$$

$$B_0^0(s=5, \omega_n) = 16\omega_n^2 - \omega_n^3$$

to calculate  $B_1^0(s=5, \omega_n=15) = 240$

$$B_0^0(s=5, \omega_n=15) = 225 \quad (4-30)$$

Using Equation (4-10) with the values of  $B_1'$ ,  $B_0'$ ,  $B_1^0$ , and  $B_0^0$  in Equations (4-29) and (4-30) and  $K_v d_1 = K_0 = 420$ , one obtains the set of solutions for the single section compensation, namely

$$p_s = -259.8$$

$$d_s = -208$$

which are not acceptable, thus the original system cannot be compensated by a single section compensator.

The original system is then cascaded with a double section (identical) compensator as

$$G(s) = \frac{420k^2(s+pk)^2}{s(s+1)(s+15)(s+p)^2}$$

The characteristic equation is then

$$F(s) = (s^5 + 16s^3 + 15s^2) + 2p(s^4 + 16s^3 + 15s^2) + p^2(s^3 + 16s^2 + 15s) + 420k^2s^2 + 840kp s + 420p^2 = 0 \quad (4-31)$$



in which  $S^2D(S) = S^5 + 16S^4 + 15S^3$

The  $B_0$  vs  $B_1$  curve for  $\zeta = .5$  associated with the polynomial  $S^2D(S)$  is, noting that the coefficient of the  $S^2$  term in  $S^2D(S)$  is zero,

$$B_1'' = -16\omega_n^3 + \omega_n^4$$

$$B_0'' = -15\omega_n^3 + \omega_n^5$$

from which

$$B_1''(\zeta=.5, \omega_n=15) = -3375$$

$$B_0''(\zeta=.5, \omega_n=15) = 708750 \quad (4-32)$$

are obtained.

Substituting the values in Equations (4-32), (4-30) and (4-29) into Equation (4-27) the locus of the  $M_d^C$  point for the compensated system with the double section compensator is defined by

$$\begin{aligned} B_1 &= -3375 + 2P(-3150) + P^2(240) + 420\alpha^2(15) \\ B_0 &= 708750 + 2P(-50625) + P^2(225) + 420\alpha^2(225) \end{aligned} \quad (4-33)$$

Using  $K_0 = 420$  and  $d_1 = 15$  in Equation (4-18), the locus of the  $M$  points is defined by

$$\begin{aligned} B_1 &= 15P^2 + 840\alpha P \\ B_0 &= 420P^2 \end{aligned} \quad (4-34)$$

By solving Equations (4-33) and (4-34) one obtains the set of solutions

$$P_s = 21.75$$

$$\alpha_s = 4.09$$

$$z_s = 5.32$$

which are the acceptable set. The compensated system transfer function is then

$$G(S) = \frac{7040(S + 5.32)^2}{S(S+1)(S+15)(S+21.75)^2}$$



The dominance of the roots at  $\zeta = .5$  and  $\omega_n = 15$  can be checked on the third order chart by factoring the characteristic equation of the compensated system into two polynomials  $(s^2 + 15s + 225)$  and  $F_1(s)$  where  $F_1(s)$  is a third order polynomial, or by sketching the  $\sqrt{.5}$  curve of the compensated system as shown in Fig. 4-5. Note that the roots at  $\zeta = .5$  and  $\omega_n = 15$  are not the dominant pair but all other roots have  $\zeta$  greater than .5.



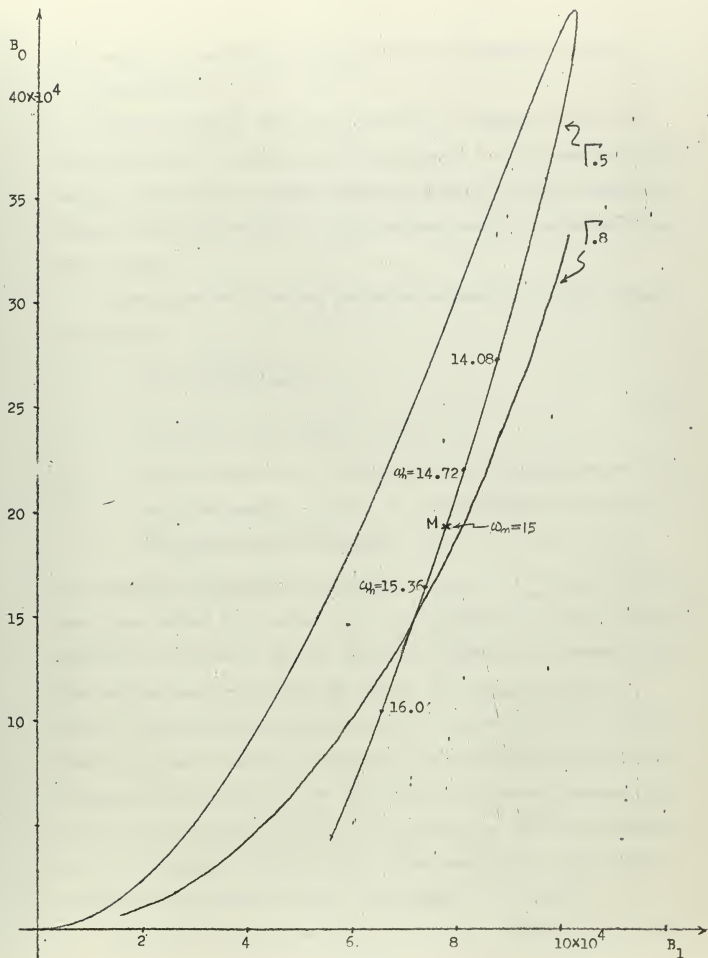


Fig. 4-5 The  $\Gamma.5$  and  $\Gamma.8$  curves for  
the compensated system. (Example 4.3)



#### 4-3. Successive application of single section compensator design procedures.

When the original system  $G_0(S)$  requires a compensator with more than one section, the system can be compensated by a non-identical multi-section compensator, by simply repeating the single section compensator design technique (Section 4-1) with judicious choice of an intermediate root location.

Returning to the equations giving the solutions of single section compensation

$$P_f = \frac{B_o'(\xi_1, \omega_{n1})}{K_o - B_o'(\xi_1, \omega_{n1})}$$

$$\alpha_f = \frac{1}{K_o} [B_i'(\xi_1, \omega_{n1}) + P_f(B_i'(\xi_1, \omega_{n1}) - d_1)] \quad (4-10)$$

where the Mitrovic  $s$  variables and the constants are defined as in Section 4-1.  $P_f$  and  $\alpha_f$  are unacceptable solutions associated with the designated root  $r_1 = -\xi_1 \omega_{n1} + \sqrt{1 - \xi_1^2} \omega_{n1}$

It is obvious in Equation (4-10) that if  $\xi_1$  and/or  $\omega_{n1}$  are changed to some other values  $\xi_1'$ , and/or  $\omega_{n1}'$  then Equation (4-10) may give some acceptable solutions  $P_1$  and  $\alpha_1$  and this is equivalent to compensating the original system to locate the root at  $r_1' = -\xi_1' \omega_{n1}' + \sqrt{1 - \xi_1'^2} \omega_{n1}'$  which is different from the designated root location  $r_1 = -\xi_1 \omega_{n1} + \sqrt{1 - \xi_1^2} \omega_{n1}$  thus the system is partially compensated by so determined single section compensator of  $P_1$  and  $\alpha_1$ . To this partially compensated system, the procedure of designing a single section compensator with the designated root  $r_1$  is applied and the system may be compensated to the desired performance requirements; if not, the procedure is repeated.

The choice of the intermediate root  $r_1'$  or the choice of  $\xi_1'$  and



$\omega_{n1}'$  is based on Equation (4-10). Since the only restriction on  $P$  is  $P > 0$ , the choice of  $\xi_1'$  and  $\omega_{n1}'$  must be in such a direction that the numerator and denominator of first of Equation (4-10) have the same signs, and yet that  $P$  with the other numbers in the second of Equation (4-10) must give an  $\alpha$  such that it is positive and not too large and not too small. Thus one must consider too many numbers simultaneously which is hard to define by a general rule. However, in view of the fact that for a smaller  $\xi$  specification the compensation is in general easier, by choosing a  $\xi$  smaller than  $\xi_1$  and analyzing the  $\Gamma_{\xi_1}^0$  and  $\Gamma_{\xi_1}'$  curves one can choose a set of suitable  $\xi_1'$  and  $\omega_{n1}'$ .

Suppose  $\xi_1'$  and  $\omega_{n1}'$  are chosen and a suitable set of  $P_1$  and  $\alpha_1$  is determined, then the partially compensated system transfer function can be written as

$$G_1(s) = \frac{K_0 \alpha_1 (s + P_1 \alpha_1)}{D(s)(s + P_1)} \quad (4-35)$$

where  $K_0/D(s) = G_0(s)$  is the original system defined as in Section 4-1,

which has the velocity constant at the specified value  $K_v$  and has a root  $r_1'$ . To compensate the system in Equation (4-35) so that it has a root  $r_1' = -\xi_1 \omega_{n1}' + \sqrt{1 - \xi_1'^2} \omega_{n1}'$  which is the specified root, a single section compensator is cascaded and the transfer function becomes

$$G(s) = \frac{K_0 \alpha_1 d(s + P_1 \alpha_1)(s + P_1 \alpha_1)}{D(s)(s + P_1)(s + P_1)} \quad (4-36)$$

The characteristic equation of the system in Equation (4-36) is then

$$F(s) = s^2 d(s) + (P_1 + P_1) s D(s) + P_1 P_1 D(s) + K_0 \alpha_1 d s^2 + K_0 \alpha_1 d (P_1 \alpha_1 + P_1 \alpha_1) s + K_0 P_1 P_1 = 0$$

$$\text{or} \quad F(s) = s^2 D(s) + P_1 s D(s) + P_1 (s D(s) + P_1 D(s)) + K_0 \alpha_1 d s^2 + (K_0 P_1 \alpha_1 + K_0 \alpha_1 P_1) s + K_0 P_1 P_1 = 0 \quad (4-37)$$



With the Mitrovic's variables for the component polynomials in Equation (4-37) defined as in Sections 4-1 and 4-2, the Mitrovic's curve for  $\xi = \xi_1$  of the compensated system in Equation (4-36) can be written in terms of the component curves, namely

$$\begin{aligned} B_1^c &= B_1''(\xi_1, \omega_n) + P B_1'(\xi_1, \omega_n) + P [B_1'(\xi_1, \omega_n) + P_1 B_1^0(\xi_1, \omega_n)] + K_o \alpha_1 \alpha \phi_2(\xi_1) \omega_n \\ B_0^c &= B_0''(\xi_1, \omega_n) + P B_0'(\xi_1, \omega_n) + P [B_0'(\xi_1, \omega_n) + P_1 B_0^0(\xi_1, \omega_n)] + K_o \alpha_1 \alpha \omega_n^2 \end{aligned} \quad (4-38)$$

By fixing  $\omega_n = \omega_{n1}$  in Equation (4-38), the locus of the  $M_d^c$  points (which is at  $\omega_n = \omega_{n1}$  on the  $\Gamma_{\xi_1}^c$  curve defined by Equation (4-38)) on the  $B_0$  vs  $B_1$  plane is defined by

$$\begin{aligned} B_1 &= B_1'' + P_1 B_1' + P (B_1' + P_1 B_1^0) + K_o \alpha_1 \alpha \phi_2(\xi_1) \omega_{n1} \\ B_0 &= B_0'' + P_1 B_0' + P (B_0' + P_1 B_0^0) + K_o \alpha_1 \alpha \omega_{n1}^2 \end{aligned} \quad (4-39)$$

where the Mitrovic's variables on the right sides of Equation (4-39) are the values of corresponding Mitrovic's equations evaluated at  $\xi = \xi_1$  and  $\omega_n = \omega_{n1}$ .

Note that in Equation (4-39) the last terms are from the component polynomial  $K_o \alpha_1 \alpha s^2$  in Equation (4-37) as in the case of the identical double section compensation in Section 4-2, and if  $\alpha_1 = \alpha$  and  $P_1 = P$  then Equation (4-39) becomes identical to Equation (4-27). Also note that in Equation (4-39), the values  $B_1^1$ ,  $B_0^1$ ,  $B_1^0$ , and  $B_0^0$  are computed at the first stage to reveal  $P_f$  and  $\alpha_f$  are unacceptable,  $B_1''$  and  $B_0''$  have to be calculated.

By collecting the  $s^1$  terms from the component polynomials in Equation (4-37) and by doing likewise for the  $s^0$  terms the locus of the  $M^c$  point on the  $B_0$  vs  $B_1$  plane for the compensated system in Equation (4-36)



is defined by

$$\begin{aligned} B_1 &= (d_1 P_1 + K_o \alpha_1) P + K_o P_1 \alpha \\ B_o &= K_o P_1 P \end{aligned} \quad (4-40)$$

Equations (4-39) and (4-40) are linear first order equations which can be solved simultaneously without difficulty to give a set of solutions in  $P$  and  $\alpha$ . If the solutions are acceptable then the compensation is accomplished, and if not then the procedures can be repeated. The following example shows the non-identical double section compensation which is seen in Section 4-2 with identical double section compensation.

#### Example 4.4

Consider the problem in Example 4.3 with the same design specifications. The original transfer function is

$$G_o(s) = \frac{420}{s(s+1)(s+15)}$$

which cannot be compensated by a single section compensator. The polynomials to be considered in the design problem are

$$D(s) = s^3 + 16s^2 + 15s$$

$$sD(s) = s^4 + 16s^3 + 15s^2$$

$$s^2D(s) = s^5 + 16s^4 + 15s^3$$

to give the associated Mitrovic's curves as follows:

$$B_1^o(\zeta, \omega_n) = 16\phi_2(\zeta)\omega_n + \phi_0\omega_n^2$$

$$B_o^o(\zeta, \omega_n) = 16\omega_n^2 - \phi_2\omega_n^3$$

$$B_1'(\zeta, \omega_n) = 15\phi_2(\zeta)\omega_n + 16\phi_3(\zeta)\omega_n^2 + \phi_4(\zeta)\omega_n^3$$

$$B_o'(\zeta, \omega_n) = 15\omega_n^2 - 16\phi_2(\zeta)\omega_n^3 - \phi_3(\zeta)\omega_n^4$$



$$B_1'(\xi, \omega_n) = 15 \phi_1(\xi) \omega_n^2 + 16 \phi_2(\xi) \omega_n^3 + \phi_3(\xi) \omega_n^4$$

$$B_0''(\xi, \omega_n) = -15 \phi_2(\xi) \omega_n^3 - 16 \phi_3(\xi) \omega_n^4 - \phi_4(\xi) \omega_n^5$$

By cascading a single section compensator to place the root at  $\xi = .5$  and  $\omega_n = 15$  resulted in an unacceptable set of solutions

$$P_f = -259.8 \qquad \alpha_f = -208$$

which indicates that the system requires a lead compensation. At the stage of getting the unacceptable solutions the following are computed;

$$B_1^0(\xi=.5, \omega_n=15) = 240$$

$$B_0^0(\xi=.5, \omega_n=15) = 225$$

$$B_1^1(\xi=.5, \omega_n=15) = -3150$$

$$B_0^1(\xi=.5, \omega_n=15) = -50625$$

Since the system cannot be compensated to have the root at  $\xi = .5$  and  $\omega_n = 15$  by using a single section compensator, an intermediate root is chosen so that the system is partially compensated by an acceptable single section compensator.

Let the intermediate root be at  $\xi = .3$  and  $\omega_n = 8$  then for this root  $B_1^0$ ,  $B_0^0$ ,  $B_1^1$  and  $B_0^1$  must be calculated. By using the corresponding equations, the following are calculated:

$$B_1^0(\xi=.3, \omega_n=8) = 117.76$$

$$B_0^0(\xi=.3, \omega_n=8) = 716.8$$

$$B_1^1(\xi=.3, \omega_n=8) = 223.6$$

$$B_0^1(\xi=.3, \omega_n=8) = -6576.6$$



By substituting those values into Equation (4-10) one obtains

$$P_1 = 22.16$$

$$\alpha_1 = 5.95, \quad z_1 = 3.73$$

which are acceptable solutions, thus the partially compensated system

$$G_1(s) = \frac{2497(s+3.73)}{s(s+1)(s+15)(s+22.16)}$$

has the root at  $\xi = .3$  and  $\omega_n = 8$ .

To set the root at  $\xi = .5$  and  $\omega_n = 15$  which is required from the design specification, another single section compensator is cascaded with the partially compensated system to give the system equation

$$G(s) = \frac{2497\alpha(s+3.73)(s+P/\alpha)}{s(s+1)(s+15)(s+22.16)(s+P)}$$

The new values to be computed are

$$B_1''(\xi=.5, \omega_n=15) = -3375$$

$$B_0''(\xi=.5, \omega_n=15) = 708750$$

By substituting  $B_1^0, B_0^0, B_1^1, B_0^1, B_1''$  and  $B_0''$  for  $\xi = .5$  and  $\omega_n = 15$  and the values of  $P_1$  and  $\alpha_1$  into Equation (4-39), one obtains the locus of the  $M_d^C$  points

$$B_1 = -3375 + (22.16)(-3150) + P(-3150 + 22.16(240)) + 2497(15)\alpha$$

$$B_0 = 70875 + (22.16)(-50625) + P(-50625 + 22.16(225)) + 2497(225)\alpha$$

or

$$B_1 = -73173 + 2168 P + 37455\alpha$$

$$B_0 = -413019 - 45639P + 561825\alpha$$

(4-41)

The locus of the  $M^C$  points of the doubly compensated system is, by using Equation (4-40),

$$B_1 = -2829P + 9306\alpha$$

$$B_0 = 9306P$$

(4-42)



Equations (4-41) and (4-42) are solved to give the solutions

$$P = 25.09$$

$$\lambda = 3.19 \quad \omega = 7.86$$

which is an acceptable set.

The compensated system is

$$G(s) = \frac{7963(s+3.73)(s+7.86)}{s(s+1)(s+15)(s+22.16)(s+25.09)}$$

and that the original system is compensated to the design specifications is confirmed by the compensated system curve  $\Gamma_5$  and the Mitrovic's working point as shown in Fig. 4-6.



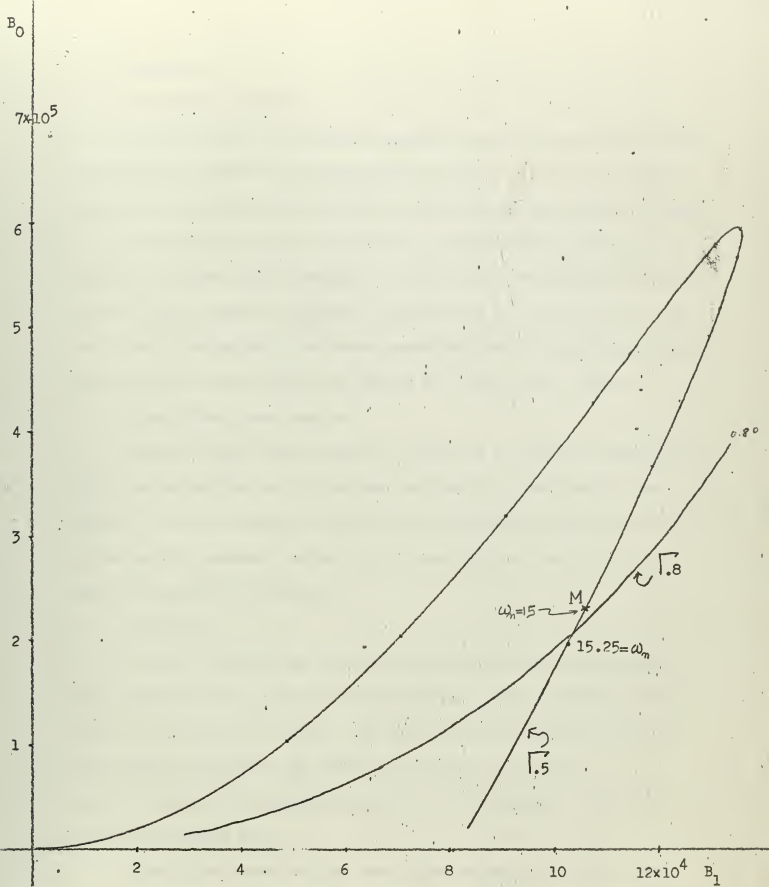


Fig. 4-6 The  $\Gamma.5$  and  $\Gamma.8$  curves for the compensated system. (Example 4.4)



## 5. Conclusion.

### 1) Frequency response.

It is seen that the frequency response of both closed and open loop systems can be obtained on the Mitrovic's  $B_0$  vs  $B_1$  plane for any order system and particularly for a system with no zeros, the method is simple.

By using the concept of the constant bandwidth curves on the  $B_0$  vs  $B_1$  plane a system can be designed so that it meets the bandwidth specification closely when the bandwidth specification is a critical one. For the third order system, a universal bandwidth chart is constructed which can be used in both analysis and design of a third order system.

### 2) The fourth order charts.

Universal fourth order charts on the  $B_0$  vs  $B_1$  plane are constructed which can be applied in both analysis and design of any fourth order system. With the preconstructed fourth order charts, the labor of constructing the necessary curves which is essential in the Mitrovic's method is removed or minimized.

### 3) Design.

An analytic method has been developed designing the required cascaded compensator by using the basic properties of the Mitrovic's equations and the  $B_0$  vs  $B_1$  curves. The method gives accurate solutions of the required compensator in order for a system to have its roots at arbitrarily designated locations on the S-plane, with arbitrarily designated gain.

If the given system and the design specifications are such that the system can be compensated by a single section compensator, the problem is to solve two linear simultaneous equations which provide solutions



quickly. For identical multi-section compensation the problem is to solve a high order polynomial in which the order of the polynomial depends upon the sections needed; however, the problem can be guided into the ones of solving linear (first order) equations by applying single section compensator design technique successively.



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# APPENDIX I

## FUNCTIONS $\phi_k(\xi)$ (GENERAL FORMULA AND TABLE)

$$\phi_k(\xi) = -(2\xi\phi_{k-1}(\xi) + \phi_{k-2}(\xi))$$

$$\phi_0(\xi) = 0$$

$$\phi_1(\xi) = -1$$

(For the proof of the formula see [1] and [2])

Table of  $\phi_k(\xi)$  functions

$\xi \backslash \phi$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	1.0
$\phi_1$	-1	-1	-1	-1	-1	-1	-1	-1	-1
$\phi_2$	0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	2.0
$\phi_3$	1.0	0.96	0.84	0.64	0.36	0	-0.44	-0.96	-3.0
$\phi_4$	0	-0.392	-0.736	-0.984	-1.088	-1.000	-0.672	-0.056	+4.000
$\phi_5$	-1.000	-0.882	-0.546	-0.050	+0.510	+1.000	+1.246	+1.038	-5.000
$\phi_6$	0	+0.568	+0.954	+1.014	+0.680	0	-0.824	-1.398	+6.000
$\phi_7$	+1.000	+0.768	+0.164	-0.559	-1.054	-1.000	-0.258	+0.918	-7.000
$\phi_8$	0	-0.722	-1.020	-0.679	+0.164	+1.000	+1.133	+0.112	+8.000
$\phi_9$	-1.000	-0.624	+0.244	+0.966	+0.923	0	-1.102	-1.075	-9.000
$\phi_{10}$	0	+0.847	+0.922	+0.099	-0.902	-1.000	+0.189	+1.393	+10.000



FORTRAN PROGRAM FOR COMPUTATION OF  $B_0$  VS  $B_1$  CURVES

(Programmed by Dr. J. R. Ward)

240



```

00390 ITEST2 = @HGRAPH
00400 ITEST3 = @HPRINT
00410 IF (ITEST1 - IOUTPUT)12,11,12
00420 INDICGR = 1
00430 GO TO 17
00440 IF (ITEST2 - IOUTPUT)14,13,14
00450 INDICGR = 0
00460 GO TO 17
00470 IF (ITEST3 - IOUTPUT)16,15,16
00480 INDICGR = 1
00490 GO TO 17
00500 IF (ITEST3 - IOUTPUT)16,15,16
00510 INDICGR = 0
00520 GO TO 17
00530 STOP
00540 FORMAT(20H ERROR IN DATA CARDS)
00550 PRINT RECORD OF INPUT DATA
00560 C 17
00570 PRINT 200
00580 PRINT 201, (JTITLE(J),J=1,7)
00590 PRINT 205, N
00600 PRINT 206, (A(J),J=2,N)
00610 PRINT 207, (A(J),J=2,N)
00620 FORMAT(1H1)
00630 FORMAT(1H1)
00640 FORMAT(4X,7A8)
00650 C 204
00660 FORMAT(10X,29H THE ORDER OF THE EQUATION IS ,I2)
00670 FORMAT(10X,39H THE COEFFICIENTS, A(2) THRU. A(N), ARE,/)
00680 C 206
00690 FORMAT(10X,11,6)
00700 C 207
00710 DETERMINING FREQUENCY RANGE OF INTEREST BY FINDING APPROXIMATE
00720 FREQUENCY AT WHICH OMEGAN**N IS LARGER THAN ANY A(J)*OMEGAN**J,
00730 J=2, THRU. N-1, 4
00740 IF (A(N)-1.0)14,10,4
00750 ANRECIP = 1./A(N)
00760 PRINT 212, 10X,45H THE RESULTS REFER TO THE NORMALIZED EQUATION. )
00770 DO 5 J=2,N
00780 A(J) = A(J)*ANRECIP
00790 OMEGAN = 0.5
00800 DC 8 K=1,12
00810 OMEGAN = OMEGAN*2.
00820 BIGTERM = 0.
00830 NM = N-1.
00840 DO 7 J=2,NM
00850 TERM = ABSF(A(J)*OMEGAN**J)
00860 IF (TERM - BIGTERM)7,7,6
00870 BIGTERM = TERM
00880 C 6
00890 CONTINUE
00900 IF (BIGTERM - OMEGAN**N)9,9,8
00910 C 8
00920 CONTINUE

```



```

C THE STEP SIZE (INCREMENT OF OMEGAN ALONG EACH CURVE) WILL BE
C TAKEN AS 0.001 TIMES THE FREQUENCY RANGE OF INTEREST
9 STEP = OMEGAN*0.001
  BOMAX = 0.
C THE MAIN COMPUTATION WILL NOW BE IN A DO LCCP, ONE PASS FOR EACH
C VALUE OF ZETA, 0.0, 0.25, 0.5, 0.75, 1.0.
DO 50 IZETA=1,5
  ZETA=IZETA-1
  ZETA=ZETA*0.25
C CALCULATE THE PHI(K) FROM EQNS. 10-12 AND 10-13 OF REF.
  PHI(1) = -1.
  PHI(2) = 2.0*ZETA
DO 21 K=3,N
  PHI(K) = -2.0*ZETA*PHI(K-1) - PHI(K-2)
C THE TERMS IN BZERO AND BONE (EGNS 10-14 AND 10-15 OF REF.) WILL
C BE COMPUTED IN TWO STAGES TO AVOID UNNECESSARY MULTIPLICATIONS.
DO 22 K=2,N
  BOMPHI(K) = A(K)*PHI(K-1)
  B1APHI(K) = A(K)*PHI(K)
DO 23 K=1,N
  OMEGAN = 1.900
  BONE = BONE + B1APHI(K)*OMEGAN*(K-1)
  OMEGAN = OMEGAN*STEP
  BZERO = 0.
DO 23 K=2,N
  BZERO = BZERO - BOMPHI(K)*OMEGAN**K
  BONE = BONE + B1APHI(K)*OMEGAN*(K-1)
C PREPARE THE ARRAYS B0 AND B1 FOR GRAPH.
  B1(OMEGAN) = BONE
  B0(OMEGAN) = BZERO
C PRINT EVERY TENTH POINT IF PRINT-OUT IS REQUIRED.
  IF(I)CICPR = 1)31,24,31
  IF(1)OMEGAN = 1)25,27,25
24 IF(X)MCDF(OMEGAN, 1)CC)26,29,26
25 IF(X)CCDF(OMEGAN, 1)CC)31,30,31
26 IF(X)CCDF(OMEGAN, 1)CC)31,30,31
27 PRINT 209, ZETA
  GO TO 21
29 PRINT 211
30 PRINT 211, OMEGAN, BZERO, BONE
209 FORMAT(1CX,35HRESULTS OF COMPUTATION WITH ZETA = ,F4.2)
210 FORMAT(1CX,35HRES, 3X, 10H BZERO, 3X, 10H BONE
211 FORMAT(1CX,35H, 3X, 10H OMEGAN, 3X, 10H, 0.0000E+00, 3X, 10H, 0.0000E+00)
C STOP COMPUTATION IF BONE OR BZERO GOES NEGATIVE, OR IF ANY
C CURVE TENDS TO GO OFF THE GRAPH. (GRAPH SCALES ARE SET AUTOMAT-
C ICALLY FOR THE ZETA = 0.00 CURVE.)

```

```

0089C
00900
00910
00920
00940
00950
00970
00990
01000
01010
01020
01030
01040
01050
01060
01070
01080
01090
01100
01110
01120
01130
01140
01150
01160
01170
01180
01190
01200
01210
01220
01230
01240
01250
01260
01270
01280
01290
01300
01310
01320
01330
01340
01350
01360
01370

```



```

31 IF(BONE)48,48,1
32 IF(ZERO)48,48,1
33 IF(IZETA-B1MAX)44,44,43
34 B1MAX=BONE
35 IF(BZERO-BONE)310,310,45
36 BOMAX=BZERO
37 GO TO 31C
38 IF(BONE-B1MAX)47,48,48
39 IF(BZERO-BOMAX)310,48,48
40 NUMPTS=(OMEGA*49)/50
41 GO TO 39
310 CONTINUE
42 NUMPTS=900
43 GRAPH IF GRAPH IS CALLED FOR.
44 IF(INCICOR-1)50,32,50
45 GO TO (33,34,35,36,37),IZETA
46 MODCURV=1
47 LABEL=4H0.00
48 GO TO 38
34 MODCURV=2
49 LABEL=4H0.25
50 GO TO 38
35 MODCURV=3
51 LABEL=4HC.5C
52 GO TO 38
36 MODCURV=4
53 LABEL=4H0.75
54 GO TO 38
37 MODCURV=5
55 LABEL=4H1.0C
56 SFX=0.
57 SFY=0.
58 MINOFFX=0
59 MINOFFY=0
60 LABELN=11
61 MODE=0
62 N1=C
63 N2=C
64 CALL GRAPH2(NUMPTS,B1,B0,8,MODCURV,LABEL,ITITLE,SFX,SFY,
MINOFFX,MINOFFY,LABELN,MCDE,N1,N2)
50 CONTINUE
65 GO TO 1000
1060 STOP
END

```

01380  
01390

01420  
01430  
01440  
01450  
01460  
01470  
01480  
01490  
01500  
01510  
01520  
01530

01570

01690  
01700  
01710  
01720  
01730  
01740

01750  
01770







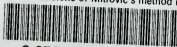






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Some extensions of Mitrovic's method in



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